

# THE CENTRALIZER OF KOMURO-EXPANSIVE FLOWS AND EXPANSIVE $\mathbb{R}^d$ -ACTIONS

WESCLEY BONOMO, JORGE ROCHA, AND PAULO VARANDAS

**ABSTRACT.** In this paper we study the centralizer of flows and  $\mathbb{R}^d$ -actions on compact Riemannian manifolds. We prove that the centralizer of every  $C^\infty$  Komuro-expansive flow with non-ressonant singularities is trivial, meaning it is the smallest possible, and deduce there exists an open and dense subset of geometric Lorenz attractors with trivial centralizer. We show that  $\mathbb{R}^d$ -actions obtained as suspension of  $\mathbb{Z}^d$ -actions are expansive if and only if the same holds for the  $\mathbb{Z}^d$ -actions. We also show that homogeneous expansive  $\mathbb{R}^d$ -actions have quasi-trivial centralizers, meaning that it consists of orbit invariant, continuous linear reparametrizations of the  $\mathbb{R}^d$ -action. In particular, homogeneous Anosov  $\mathbb{R}^d$ -actions have quasi-trivial centralizer.

## 1. INTRODUCTION

One of the leading problems considered by the dynamical systems community has been to describe the features of most dynamical systems. Based on the pioneering works of Peixoto and Smale, the program proposed by Palis in the nineties has constituted a route guide for a global itemize of the space of dynamical systems. This program, that proposed the complement of uniform hyperbolicity as the space of diffeomorphisms that are approximated by those exhibiting either heteroclinic tangencies or heteroclinic cycles, was carried out much successfully in the  $C^1$ -topology, where perturbation tools like the closing lemma, Franks' lemma, connecting lemma or ergodic closing lemma are available [27, 16].

In seminal papers, Smale [38, 39] conjectured that most dynamical systems should have trivial centralizer, a hard problem not yet completely understood. Given a  $C^r$ -diffeomorphism  $f$  on a compact manifold  $M$ , its  $C^r$ -centralizer  $\mathcal{Z}^r(f) = \{g \in \text{Diff}^r(M) : f \circ g = g \circ f\}$  is a subgroup of  $\text{Diff}^r(M)$ ,  $r \geq 1$ . In some sense, the centralizer reflects symmetries of the dynamic which typically should be rare. The problem of the centralizer is related e.g. with the embedding of maps as time-1 maps of flows [32] or the problem of differentiability of conjugacies [42]. In the discrete-time setting some results in the direction of a positive answer to Smale's conjecture include: (i) expansive homeomorphisms have discrete centralizers [41], (ii) there are open and dense subsets of  $C^r$  ( $r \geq 2$ ) circle diffeomorphisms [21], of  $C^\infty$ -Axiom A diffeomorphisms with the strong transversality property and a periodic sink [33], of  $C^\infty$ -Axiom A surface diffeomorphisms with the strong transversality condition [35], and codimension one hyperbolic attractors of  $C^r$ -diffeomorphisms [14] with trivial

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centralizers; (iii) there are open sets of surface Anosov diffeomorphisms whose  $C^0$ -centralizer is discrete but not trivial [36]; (iv) on the torus of dimension 2, 3, or 4, the subset of diffeomorphisms with trivial centralizer in the  $C^1$  topology has nonempty interior [4] (v) there exists a  $C^1$ -residual subset of  $\text{Diff}^1(M)$  with trivial centralizer [8] and the set of  $C^1$ -diffeomorphisms with trivial centralizer is not open and dense [7]; and (vi) there exists a residual subset of certain classes of  $C^r$ ,  $r \geq 1$ , partially hyperbolic diffeomorphisms with discrete centralizer [12].

In the time-continuous setting the picture is still much more incomplete. In opposition to the discrete time setting the centralizer is never discrete. Indeed, using that the flow commutes with itself, the centralizer of a flow clearly contains a continuum. Some advances to establish the counterpart of Smale's conjecture for Anosov,  $C$ -expansive and Axiom A flows with the strong transversality were obtained in [18], [29], [37] and [23]. Such a description of the centralizer can be given in terms of the flow or of the generating vector field. However, flows with some weak hyperbolicity and where singularities accumulated by regular orbits of the flow have not yet been considered, and this is a first goal of the present work. These include important classes of flows as three-dimensional  $C^1$ -robustly transitive flows with singularities, often referred as singular-hyperbolic flows. These are partially hyperbolic flows with an invariant splitting in a one-dimensional contracting and a two-dimensional volume expanding invariant subbundles for the vector field  $X$  or the vector field  $-X$ . Any singular-hyperbolic flow without singularities is an Anosov flow. Hence, if the manifold does not support Anosov flows then the singular set of a singular-hyperbolic flow is non-empty. Moreover, singular-hyperbolic attractors include the important classes of examples of geometric Lorenz attractors, introduced by Afraimovich, Bykov and Shilnikov [1] and Guckenheimer, Williams [15] to model the chaotic attractor proposed by E. Lorenz [22]. We refer the reader to [2] for precise definitions and a large account on singular-hyperbolicity.

Our first purpose in the current article is to describe the centralizer of a class of flows that contains the important classes of geometric Lorenz attractors. These admit a weak form of expansiveness (the so called Komuro-expansiveness) which is compatible with the coexistence of regular and singular orbits in the same transitive piece of the non-wandering set. We prove first that Komuro-expansive flows have quasi-trivial centralizers: any commuting flow is a continuous linear reparametrization of the original flow. If, in addition, the singularities satisfy an (open and dense) non-ressonance condition then the centralizer of such a vector field  $X$  consists of the vector fields of the form  $cX$  for some  $c \in \mathbb{R}$  on the closure of the stable manifolds of the singularities (cf. Theorem A and Corollary A). As a byproduct of these results we conclude that the orbits of  $\mathbb{R}^d$ -actions that admit some expansive element are indeed one-dimensional, a fact that holds e.g. for Anosov actions (cf. Corollary B). As a second purpose we also obtain a systematic treatment of the centralizer of expansive  $\mathbb{R}^d$ -actions. In the case of  $\mathbb{R}^d$ -actions that are suspension of  $\mathbb{Z}^d$ -actions, expansiveness is either a common feature or it fails for both actions (cf. Theorem C). We also prove that the centralizer of expansive "typical homogeneous"  $\mathbb{R}^d$ -actions is also reduced to its continuous reparametrizations. We refer the reader to Theorem B for the precise statement.

This paper is organized as follows. In Section 2 we introduce some definitions and state the main results of this paper. Section 6 is devoted to present some examples and a wider discussion and comparison of our results with other notions

of expansiveness for flows. In Section 3 we study the centralizer of expansive flows with singularities. The results on the expansiveness properties and centralizer of  $\mathbb{R}^d$ -actions are given along Sections 4 and 5.

## 2. PRELIMINARIES AND STATEMENT OF THE MAIN RESULTS

**2.1. Preliminaries.** In this subsection we shall introduce some definitions and recall some necessary background, with the intention of making the text as self-contained as possible. Throughout we let  $M$  be a compact Riemannian manifold.

**2.1.1. Uniform hyperbolicity.** Given a  $C^r$ -diffeomorphism  $f$  on  $M$ ,  $r \geq 1$ , a set  $\Lambda \subset M$  is *uniformly hyperbolic* if there is a  $Df$ -invariant splitting  $T_\Lambda M = E^s \oplus E^u$  and constants  $C > 0$  and  $\lambda \in (0, 1)$  so that

$$\|Df^n(x)|_{E_x^s}\| \leq C\lambda^n \quad \text{and} \quad \|(Df^n(x)|_{E_x^u})^{-1}\| \leq C\lambda^n$$

for every  $x \in \Lambda$  and  $n \geq 1$ . We refer to  $T_\Lambda M = E^s \oplus E^u$  as the *hyperbolic splitting* associated to  $f$  and  $\Lambda$ .

Let  $\text{Per}(f)$  denote the set of periodic points and  $\Omega(f)$  denote the non-wandering set of  $f$ . A  $C^1$ -diffeomorphism  $f$  is called *Axiom A* if  $\overline{\text{Per}(f)} = \Omega(f)$  and  $\Omega(f)$  is a uniformly hyperbolic set. The diffeomorphism  $f$  is *Anosov* if the manifold  $\Lambda = M$  is a hyperbolic set for  $f$ .

The natural counterpart for flows is defined as follows. Given a vector field  $X \in \mathfrak{X}^r(M)$ ,  $r \geq 1$ , let  $(\varphi_t)_{t \in \mathbb{R}}$  denote the  $C^r$ -flow on  $M$  generated by  $X$ . Recall that  $\sigma \in M$  is a *hyperbolic singularity* for  $X \in \mathfrak{X}^1(M)$  provided  $X(\sigma) = 0$  and  $DX(\sigma)$  does not contain any purely imaginary eigenvalue. Furthermore, we say that a hyperbolic singularity  $\sigma$  for  $X \in \mathfrak{X}^1(M)$  is *non-ressonant* if the eigenvalues  $\alpha_1, \dots, \alpha_k \in \mathbb{C}$  of  $DX(\sigma)|_{E_\sigma^u}$  (resp. the eigenvalues  $\beta_1, \dots, \beta_m \in \mathbb{C}$  of  $DX(\sigma)|_{E_\sigma^s}$ ) are all distinct and do not satisfy any relation of the form  $\text{Re}(\alpha_i) = \sum_{j \neq i} n_j \text{Re}(\alpha_j)$  (resp.  $\text{Re}(\beta_i) = \sum_{j \neq i} n_j \text{Re}(\beta_j)$ ) for some non-negative integers  $n_j$  so that  $\sum_{j=1}^k n_j \geq 2$ . Observe that since we consider the eigenvalues of stable and unstable bundles independently, the later corresponds the singularity to be separately non-ressonant. Furthermore, in this case the resonance conditions consist of *finitely* many algebraic closed equations and, consequently, are satisfied by an open and dense subset of linear vector fields.

Given a compact  $(\varphi_t)_{t \in \mathbb{R}}$ -invariant non-singular set  $\Lambda \subset M$ , we say that  $\Lambda$  is a *hyperbolic set* for  $(\varphi_t)_{t \in \mathbb{R}}$  if there exists a  $D\varphi_t$ -invariant splitting  $T_\Lambda M = E^s \oplus E^0 \oplus E^u$  so that: (a)  $E^0$  is one dimensional and generated by the vector field, (b) there are constants  $C > 0$  and  $\lambda \in (0, 1)$  so that

$$\|D\varphi_t(x)|_{E_x^s}\| \leq C\lambda^t \quad \text{and} \quad \|(D\varphi_t(x)|_{E_x^u})^{-1}\| \leq C\lambda^t$$

for every  $x \in \Lambda$  and  $t \geq 0$ .

Given a  $C^1$ -flow  $\varphi = (\varphi_t)_t$  we denote by  $\text{Sing}(\varphi)$  the singularities of  $\varphi$  and by  $\text{Crit}(\varphi)$  the set of all critical elements, formed by singularities and closed orbits for the flow  $\varphi$ . A flow  $(\varphi_t)_t$  is called *Axiom A* if  $\overline{\text{Crit}(\varphi)} = \Omega(X)$  and the non-wandering set  $\Omega(X)$  is a uniformly hyperbolic set. The flow  $(\varphi_t)_t$  is *Anosov* if  $\Lambda = M$  is a hyperbolic set.

We say that  $\Phi : \mathbb{R}^d \times M \rightarrow M$  is a  $C^r$ -action on a compact Riemannian manifold  $M$  if  $\Phi_v := \Phi(v, \cdot) : M \rightarrow M$  is a  $C^r$  diffeomorphism and  $\Phi_{v+u} = \Phi_v \circ \Phi_u$  for every  $v, u \in \mathbb{R}^d$ . Following [6], we say that a  $C^r$ -action  $\Phi : \mathbb{R}^d \times M \rightarrow M$  on a compact Riemannian manifold  $M$  is an *Anosov action* if there exists  $v \in \mathbb{R}^d$

such that the diffeomorphism  $\Phi_v$  admits a continuous  $D\Phi_v$ -invariant decomposition  $TM = E_v^s \oplus T\Phi \oplus E_v^u$  where  $T\Phi$  denotes the tangent space to the orbits of  $\Phi$  and there are constants  $C > 0$  and  $\lambda \in (0, 1)$  so that

$$\|D\Phi_v^n(x) |_{E_x^s}\| \leq C\lambda^n \quad \text{and} \quad \|(D\Phi_v^n(x) |_{E_x^u})^{-1}\| \leq C\lambda^n$$

for every  $x \in M$  and  $n \geq 0$ . The diffeomorphism  $\Phi_v$  is called an *Anosov element*.

Let  $\mathcal{F}$  denote the orbit foliation of  $\Phi$  and let  $\mathcal{F}(x)$  denote the leaf of the foliation containing the point  $x$ . It follows from [17, Theorem 7.2] that  $(\Phi_v, \mathcal{F})$  is a plaque expansive diffeomorphism: there exists  $\bar{\delta} > 0$  such that if  $(x_n)_{n \in \mathbb{Z}}$  and  $(y_n)_{n \in \mathbb{Z}}$  are  $\bar{\delta}$ -pseudo-orbits preserving  $\mathcal{F}$  (ie.  $d(\Phi_v(x_n), x_{n+1}) < \bar{\delta}$ ,  $d(\Phi_v(y_n), y_{n+1}) < \bar{\delta}$ ,  $\Phi_v(x_n) \in \mathcal{F}_\delta(x_{n+1})$  and  $\Phi_v(y_n) \in \mathcal{F}_\delta(y_{n+1})$  for all  $n \in \mathbb{Z}$ ) and the pair of points  $x_n, y_n$  remains  $\bar{\delta}$ -close for all  $n$ , then  $y_n \in \mathcal{F}_\delta(x_n)$  for every  $n \in \mathbb{Z}$ .

**2.1.2. Expansiveness.** First we shall recall the notion of expansiveness in the discrete time setting. Given a homeomorphism  $f \in \text{Homeo}(M)$  and a compact invariant set  $\Lambda \subset M$ , we say that  $f$  is *expansive* in  $\Lambda$  if there exists  $\delta > 0$  so that for all  $x, y \in \Lambda$  satisfying  $d(f^n(x), f^n(y)) \leq \delta$  for every  $n \in \mathbb{Z}$  one has  $x = y$ . In the time-continuous setting of flows, due to the possible presence of singularities, there are several notions of expansiveness (see e.g. [11, 30]). We recall some of these notions, starting by the one introduced by Bowen and Walters [11].

**Definition 2.1.** Let  $(M, d)$  be a compact metric space,  $\varphi : \mathbb{R} \times M \rightarrow M$  be a continuous flow, and  $\Lambda \subseteq M$  be a compact  $\varphi$ -invariant set. We say that the flow  $\varphi$  is *C-expansive* in  $\Lambda$  if for any  $\varepsilon > 0$  there exists  $\delta > 0$  so that if  $x, y \in \Lambda$  and  $d(\varphi_t(x), \varphi_{h(t)}(y)) < \delta$  for all  $t \in \mathbb{R}$  for some continuous function  $h : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $h(0) = 0$ , then  $y = \varphi_{t_0}(x)$  for some  $|t_0| < \varepsilon$ .

The later means that, for expansive flows, orbits of two points  $x, y$  by the flow that always remain close to each other (up to reparametrization) do coincide. Singularities of a *C-expansive* flow are necessarily isolated points [11, Lemma 1]. Moreover, *C-expansive* flows on connected manifolds do not admit singularities. A weaker notion, as follows, was introduced later by Keynes and Sears [19].

**Definition 2.2.** Let  $(M, d)$  be a compact metric space,  $\varphi : \mathbb{R} \times M \rightarrow M$  be a continuous flow, and  $\Lambda \subseteq M$  be a compact  $\varphi$ -invariant set. We say that the flow  $\varphi$  is *K-expansive* in  $\Lambda$  if for any  $\varepsilon > 0$  there exists  $\delta > 0$  so that if  $x, y \in \Lambda$  and  $d(\varphi_t(x), \varphi_{h(t)}(y)) < \delta$  for all  $t \in \mathbb{R}$  for some increasing homeomorphism  $h : \mathbb{R} \rightarrow \mathbb{R}$  with  $h(0) = 0$  then  $y = \varphi_{t_0}(x)$  for some  $|t_0| < \varepsilon$ .

Although the later is more general, these two notions are indeed equivalent in the case that  $M$  is a compact Riemannian manifold (see e.g. [2]). Motivated by the analysis of flows with non-isolated singularities in the non-wandering set as the classical geometric Lorenz attractors, Komuro [20] introduced a more general notion of expansiveness that we now describe.

**Definition 2.3.** Let  $(M, d)$  be a compact metric space,  $\varphi : \mathbb{R} \times M \rightarrow M$  be a continuous flow, and  $\Lambda \subseteq M$  be a compact  $\varphi$ -invariant set. We say that the flow  $\varphi$  is *Komuro-expansive* in  $\Lambda$  if for any  $\varepsilon > 0$  there exists  $\delta > 0$  so that if  $x, y \in \Lambda$  and  $d(\varphi_t(x), \varphi_{h(t)}(y)) < \delta$  for every  $t \in \mathbb{R}$  and for some increasing homeomorphism  $h : \mathbb{R} \rightarrow \mathbb{R}$  then there is  $t_0 \in \mathbb{R}$  such that  $\varphi_{h(t_0)}(y) \in \varphi_{[t_0-\varepsilon, t_0+\varepsilon]}(x)$ . Here, as usual,  $\varphi_{[t_0-\varepsilon, t_0+\varepsilon]}(x) := \{\varphi_t(x) : t \in [t_0 - \varepsilon, t_0 + \varepsilon]\}$ .

Sometimes Komuro-expansive flows are simply called expansive in literature, and we keep this nomenclature. It follows from the previous definitions that

$$C\text{-expansiveness} \Rightarrow K\text{-expansiveness} \Rightarrow \text{expansiveness}. \quad (2.1)$$

On the converse direction, these three notions of expansiveness are equivalent for flows without singularities (cf. [30, Theorem A]) but may differ for flows with singularities. If not explicated otherwise we assume  $\Lambda = M$ , meaning expansiveness in the whole manifold. It is well known that uniformly hyperbolic flows are  $C$ -expansive and, for that reason, expansive diffeomorphisms and flows contain as special classes of examples the case Anosov diffeomorphisms and Anosov flows, respectively.

In the case of  $\mathbb{R}^d$ -actions, each orbit of a point  $x$  in the manifold  $M$  is an immersed submanifold of dimension at most  $d$ . In the case that the foliation formed by the orbits of points consist of submanifolds with different dimensions, this encloses the same kind of topological difficulties (in a wider range of possibilities) caused by the presence of singularities for flows.

**Definition 2.4.** Let  $M$  be a compact metric space and  $\Phi : \mathbb{R}^d \times M \rightarrow M$  be a continuous action. We say that  $\Phi$  is expansive if for any  $\varepsilon > 0$ , there exists  $\delta > 0$  so that if  $x, y \in M$  satisfy  $d(\Phi_v(x), \Phi_{h(v)}(y)) < \delta$  for every  $v \in \mathbb{R}^d$  with respect to a continuous function  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$  so that  $h(0) = 0$ , then  $y = \Phi_{v_0}(x)$  for some  $\|v_0\| < \varepsilon$ . In particular,  $y$  belongs to the orbit of  $x$  by  $\Phi$ .

The study of the geometry and topology of the foliation by orbits of  $\mathbb{R}^d$ -actions is a hard problem and encloses much more difficulties than the case of flows. For instance, in opposition to the case of vector fields, we do not expect all expansive  $\mathbb{R}^d$ -actions on compact connected Riemannian manifolds to be homogeneous.

**2.1.3. Centralizers.** Given  $r \geq 0$  and a diffeomorphism  $f \in \text{Diff}^r(M)$  the *centralizer* of  $f$  is the set

$$\mathcal{Z}^r(f) = \{g \in \text{Diff}^r(M) : f \circ g = g \circ f\}$$

where, by some abuse of notation, we let  $\text{Diff}^0(M)$  denote the space of homeomorphisms. The definition for time-continuous dynamical systems is analogous. Given  $r \geq 0$ , let  $\mathcal{F}^r(M)$  denote the space of  $C^r$ -flows on  $M$ . Given a flow  $\varphi = (\varphi_t)_{t \in \mathbb{R}} \in \mathcal{F}^r(M)$ , the *centralizer* of  $\varphi$  is defined as

$$\mathcal{Z}^r(\varphi) = \{\psi = (\psi_t)_{t \in \mathbb{R}} \in \mathcal{F}^r(M) : \psi_s \circ \varphi_t = \varphi_t \circ \psi_s, \forall s, t \in \mathbb{R}\}.$$

It is clear from the previous definition that flows obtained as reparametrizations of a flow  $\varphi$  belong to  $\mathcal{Z}^r(\varphi)$ . For that reason, the centralizer of a flow is never a discrete subgroup. In the case of smooth flows, the previous characterization of centralizer has a dual formulation in terms of vector fields. Given  $r \geq 1$  and  $X \in \mathfrak{X}^r(M)$ , one can define the *centralizer of the vector field*  $X$  by

$$\mathcal{Z}^r(X) = \{Y \in \mathfrak{X}^r(M) : [X, Y] = L_Y X = 0\},$$

where  $L_Y X$  denotes the Lie derivative of the vector field  $X$  along  $Y$ .

**Definition 2.5.** Given  $r \geq 0$ , we say a flow  $\varphi = (\varphi_t)_t \in \mathcal{F}^r(M)$  has quasi-trivial centralizer if for any  $\psi \in \mathcal{Z}^r(\varphi)$  there exists a  $C^r$ -function  $A : M \rightarrow \mathbb{R}$  so that

- (i) (orbit invariance)  $A(x) = A(\varphi_t(x))$  for every  $(t, x) \in \mathbb{R} \times M$ , and
- (ii)  $\psi_t(x) = \varphi_{A(x)t}(x)$  for every  $(t, x) \in \mathbb{R} \times M$ .

In the case that the reparametrizations  $A$  are necessarily constant then we say the centralizer is trivial. Dually, we say that  $X \in \mathfrak{X}^r(M)$  has quasi-trivial centralizer if for any  $Y \in \mathcal{Z}^r(X)$  there exists  $A \in C^r(M, \mathbb{R})$  so that  $Y = A \cdot X$  and  $X(A) = 0$ . If  $A$  is constant then we say that the centralizer is trivial.

Observe that the previous notions for vector fields and flows are dual. On the one hand, if  $X(h) = 0$  for some  $h : M \rightarrow \mathbb{R}$  then  $h$  is constant along the orbits of the flow. On the other hand, if  $Y = h \cdot X$  and  $h$  is constant along the orbits of the flow  $(X_t)_t$  generated by  $X$  then the flow  $(Y_t)_t$  generated by  $Y$  satisfies  $Y_t(x) = X_{h(x)t}(x)$  for every  $t \in \mathbb{R}$  and  $x \in M$ . The centralizer of a  $\mathbb{R}^d$ -action is defined similarly.

**Definition 2.6.** Given a  $C^r$ -action  $\Phi : \mathbb{R}^d \times M \rightarrow M$ , we define its centralizer as the set  $\mathcal{Z}^r(\Phi) = \{\Psi : \mathbb{R}^d \times M \rightarrow M : \Phi_v \circ \Psi_u = \Psi_u \circ \Phi_v \text{ for all } v, u \in \mathbb{R}^d\}$ . We say that  $\Phi$  has a quasi-trivial centralizer if for any  $\Psi \in \mathcal{Z}^1(\Phi)$  there exists a  $C^r$ -map  $A : M \rightarrow \mathcal{M}_{d \times d}(\mathbb{R})$  satisfying  $A(x) = A(\Phi_v(x))$  for every  $v \in \mathbb{R}^d$  and  $x \in M$  and so that  $\Psi_v(x) = \Phi(A(x)v, x)$  for every  $(v, x) \in \mathbb{R}^d \times M$ .

**2.2. Statement of the main results.** This subsection is devoted to the statement of the main results.

**2.2.1. Centralizers of Komuro-expansive flows.** It is known that  $C$ -expansive flows on compact and connected metric spaces have quasi-trivial centralizer [29] (the author used the nomenclature of “unstable centralizer”). Our first result is an extension of the aforementioned results for the broader class of expansive flows.

**Theorem A.** Let  $\varphi$  be a  $C^\infty$  flow on a compact, connected Riemannian manifold  $M$  and let  $\Lambda \subset M$  be a compact  $\varphi$ -invariant set such that  $\varphi$  is transitive and Komuro-expansive in  $\Lambda$ . If all the singularities of  $\varphi|_\Lambda$  are hyperbolic and non-ressonant then the centralizer  $\mathcal{Z}^\infty(\varphi|_\Lambda)$  is quasi-trivial. Thus, for any  $\psi \in \mathcal{Z}^\infty(\varphi|_\Lambda)$  there exists a  $C^\infty$ -map  $A : \Lambda \rightarrow \mathbb{R}$ , constant along the orbits of  $\varphi|_\Lambda$  (meaning  $A(x) = A(\varphi_t(x))$  for every  $x \in \Lambda$  and  $t \in \mathbb{R}$ ) so that  $\psi_t(x) = \varphi_{A(x)t}(x)$  for every  $(t, x) \in \mathbb{R} \times \Lambda$ .

Some comments are in order. Firstly, the  $C^\infty$  regularity assumption in the previous theorem is not used in full strength. Indeed, the argument in the proof of the theorem can be divided two main steps: (i) the orbit of a regular point by an element in the centralizer is a reparametrization of the original trajectory, and (ii) the reparametrization obtained at regular orbits extend continuously to singularities.

Our strategy combines the linearization at hyperbolic singularities (which requires sufficiently regularity of the vector field given in terms of conditions on the eigenvalues of the singularities as in Sternberg’s linearization results) together with the characterization due to Kopell [21] that the  $C^r$ -centralizer of their linear part is formed only by linear transformations provided that  $r$  is large enough (we refer the reader to Subsection 3.5 for the details). The  $C^\infty$  assumption allows to simplify the statement. Moreover, the centralizer can be proved trivial in the case that stable/unstable manifolds of singularities are dense in the phase space. Secondly, we observe that a version of Theorem A for volume preserving flows also holds. This follows straightforwardly from the arguments used in the proof of Theorem A using linearization results for volume preserving vector fields (see e.g. [5] and references therein). Thirdly, Komuro-expansive flows do not form a  $C^1$ -open set of flows. Nevertheless, the  $C^1$ -interior of the set of Komuro-expansive



flows is not empty and contains the important classes of Anosov and singular-hyperbolic flows. Since our results apply to proper invariant sets, in particular we deduce there exists an open and dense subset of  $C^\infty$ -geometric Lorenz attractors with trivial centralizer (cf. Example 6.2). Finally, recalling the duality between commuting flows and vector fields, Theorem A admits the following reformulation: if  $X \in \mathfrak{X}^\infty(M)$  generates a Komuro-expansive flow on a compact and connected Riemannian manifold  $M$  so that all the singularities for  $X$  are hyperbolic and non-ressonant, then for any  $Y \in \mathcal{Z}^\infty(X)$  there exists  $h \in C^0(M, \mathbb{R})$  so that  $Y = h \cdot X$  and  $X(h) = 0$ .

Our strategy implies on the quasi-triviality of the centralizer on the topological basin of attraction of attractors. Using spectral decomposition in finitely many basic pieces, Sad [37, Theorem B] proved that there is an open and dense subset  $A'_\tau$  of the  $C^\infty$ -Axiom A vector fields with the strong transversality condition so that  $\mathcal{Z}^\infty(X) = \{cX : c \in \mathbb{R}\}$  for every  $X \in A'_\tau$ . The following can be understood as an extension of [37], where expansiveness and the (open and dense) non-ressonance condition replaces the uniformly hyperbolic assumption of [37].

**Corollary A.** *Let  $\varphi$  be an expansive  $C^\infty$ -flow on a compact and connected Riemannian manifold  $M$ . Assume that all the singularities  $\text{Sing}(\varphi)$  are hyperbolic and non-ressonant. If*

$$\Lambda = \overline{\bigcup_{\sigma \in \text{Sing}(\varphi)} W^s(\sigma)} \quad \left( \text{or } \Lambda = \overline{\bigcup_{\sigma \in \text{Sing}(\varphi)} W^u(\sigma)} \right)$$

*then  $\mathcal{Z}^\infty(\varphi|_\Lambda)$  is trivial. In other words, if  $\psi \in \mathcal{Z}^\infty(\varphi|_\Lambda)$  then there exists  $c \in \mathbb{R}$  so that  $\psi_t(x) = \varphi_{ct}(x)$  for every  $t \in \mathbb{R}$  and  $x \in \Lambda$ .*

**2.2.2. Centralizers of  $\mathbb{R}^d$ -actions.** Our previous results have implications for the study of the centralizer of smooth  $\mathbb{R}^d$ -actions that admit expansive elements. For example, the following is a consequence of Theorem A.

**Corollary B.** *Take  $d \geq 1$  and let  $\Phi : \mathbb{R}^d \times M \rightarrow M$  be a continuous  $\mathbb{R}^d$ -action on a compact Riemannian manifold  $M$ . If there exists  $v \in \mathbb{R}^d$  so that  $(\Phi_{tv})_{t \in \mathbb{R}}$  is an expansive flow then the orbits of  $\Phi$  are one-dimensional and coincide with the orbits of a flow.*

By the previous corollary, if an  $\mathbb{R}^d$ -action has some orbit of dimension larger than one then there exists no  $v \in \mathbb{R}^d$  so that  $(\Phi_{tv})_{t \in \mathbb{R}}$  is an expansive flow. In what follows we shall introduce the notion of homogeneous  $\mathbb{R}^d$ -actions.

**Definition 2.7.** *Let  $M$  be a compact Riemannian manifold. We say that the  $\mathbb{R}^d$ -action  $\Phi : \mathbb{R}^d \times M \rightarrow M$  is homogeneous if all orbits by  $\Psi$  are submanifolds on  $M$  with dimension equal to  $d$ .*

We observe that a flow is homogeneous if and only it has no singularities. In particular, not every manifold admits homogeneous  $\mathbb{R}^d$ -actions (e.g. every  $C^1$ -flow on  $\mathbb{S}^2$  admits singularities, by Poincaré-Bendixson theorem). On the other hand, manifolds that support homogeneous  $\mathbb{R}^d$ -actions include the torus  $\mathbb{T}^n$  ( $n \geq d$ ) (using suspension actions defined in Section 4), and the space of homogeneous  $\mathbb{R}^d$ -actions forms a open subset of all  $\mathbb{R}^d$ -actions. Indeed, if  $(e_1, \dots, e_d)$  denotes a basis of  $\mathbb{R}^d$ ,

$\Phi : \mathbb{R}^d \times M \rightarrow M$  is a smooth  $\mathbb{R}^d$ -action, and  $\varphi_{e_i} := (\Phi_{te_i})_{t \in \mathbb{R}}$  denotes the canonical flow generated by the direction  $e_i$

$$\begin{aligned} \varphi_{e_i} : \mathbb{R} \times M &\rightarrow M \\ (t, x) &\mapsto \Phi(te_i, x) \end{aligned}$$

then it is not hard to check that  $\Phi$  is homogeneous if and only if the vector fields  $X_{e_i}(x) = \frac{d}{dt}\varphi_{e_i}(t, x)|_{t=0}$  are linearly independent at all points of  $M$ , which is clearly an open condition. In what follows we describe the centralizer of expansive and homogeneous  $\mathbb{R}^d$ -actions.

**Theorem B.** *Let  $M$  be a compact Riemannian manifold and  $\Phi : \mathbb{R}^d \times M \rightarrow M$  be a continuous action. If  $\Phi$  is expansive and homogeneous then  $\mathcal{Z}^1(\Phi)$  is quasi-trivial, i.e., there exists a  $C^1$ -map  $A : M \rightarrow \mathcal{M}_{d \times d}(\mathbb{R})$  satisfying  $A(x) = A(\Phi_v(x))$  for every  $v \in \mathbb{R}^d$  and so that  $\Psi_v(x) = \Phi(A(x)v, x)$  for every  $(v, x) \in \mathbb{R}^d \times M$ .*

The previous result is the counterpart of [29] for  $\mathbb{R}^d$ -actions. We also obtain a geometrical interpretation for the reparametrization  $A(\cdot)$  obtained above (see Proposition 5.1) and prove that Anosov actions also have quasi-trivial centralizer (see Example 6.6). The description of the centralizer of non-homogeneous, expansive  $\mathbb{R}^d$ -actions encodes difficulties similar to the ones for flows where singular and non-singular orbits coexist. The strategy used in the case of flows with singularities can probably be applied similarly in the case that the set of singular orbits (ie. of dimension smaller than  $d$ ) has empty interior in  $M$ . More generally, it is not hard to see that elements in the centralizer of any  $\mathbb{R}^d$ -action preserve orbits of the same dimension but it is unclear if these are reparametrizations of the original action. We also establish a characterization of expansive  $\mathbb{R}^d$ -actions obtained as suspensions of  $\mathbb{Z}^d$ -actions (Theorem C). Since the precise statement of this result requires many extra definitions we will state and prove it in Section 4).

### 3. CENTRALIZER OF EXPANSIVE FLOWS

This section is devoted to the proof of Theorem A. We subdivide the proof in subsections for making the exposition clearer. Let  $\varphi$  be a  $C^\infty$  flow defined in a compact manifold  $M$  and  $\Lambda \subset M$  be a compact  $\varphi$ -invariant subset on which the flow is expansive and all singularities are hyperbolic and non-ressonant. First we prove that all periodic orbits on  $\Lambda$ , if they exist, have their periods larger than some uniform lower bound (Lemma 3.2). This is enough to obtain tubular flowboxes of uniform size  $\mu$ , at each regular point, and to use expansiveness to guarantee that every flow  $\psi \in \mathcal{Z}^\infty(\varphi|_\Lambda)$  is locally a reparametrization of the flow  $\varphi$  on  $\Lambda \setminus \text{Sing}(\varphi|_\Lambda)$  and that such local reparametrization is unique and defined for all time in  $[-\mu, \mu]$  (cf. Lemma 3.3). Then we borrow the strategy of [29] to prove that although  $\Lambda \setminus \text{Sing}(\varphi|_\Lambda)$  may be non-compact the local reparametrizations of the orbits can be uniquely extended to the real line  $\mathbb{R}$  on all regular points  $x \in \Lambda \setminus \text{Sing}(\varphi|_\Lambda)$  (cf. Proposition 3.1). Finally, we show that the reparametrizations are linear and constant along orbits of regular points of  $\varphi|_\Lambda$ . Using the linearization of the singularities together with a version of Kopell's description of the centralizer of linear flows (Lemma 3.5) we conclude that the reparametrizations can be continuously extended to the singularities (hence, are globally defined in  $\Lambda$ ) and that these are smooth. Throughout this section, and for notational simplicity, we use the notation  $\varphi_t$  instead of  $(\varphi|_\Lambda)_t$ .



**3.1. Bound on length of periodic orbits.** In next lemma we prove the existence of positive infimum for the period of periodic orbits of regular points of  $\varphi|_\Lambda$ , in case they exist. We will need the tubular neighborhood theorem for vector fields.

**Lemma 3.1.** *Let  $M$  be a compact Riemannian manifold of dimension  $n$ . Given  $X \in \mathfrak{X}^1(M)$  and a regular point  $x \in M$  there exists  $\delta = \delta_x > 0$ , an open neighborhood  $U_x^\delta$  of  $x$  (called tubular neighborhood), and a  $C^1$ -diffeomorphism  $\Psi_x : U_x^\delta \rightarrow (-\delta, \delta) \times B(x, \delta) \subset \mathbb{R} \times \mathbb{R}^{d-1}$  such that the vector field  $X$  on  $U_x^\delta$  is the pull-back of the vector field  $Y := (1, 0, \dots, 0)$  on  $(-\delta, \delta) \times B(x, \delta)$ , that is,  $Y = (\Psi_x)_* X := D(\Psi_x)_{\Psi_x^{-1}} X(\Psi_x^{-1})$ . In particular  $Y_t(\cdot) = \Psi_x(X_t(\Psi_x^{-1}(\cdot)))$  for every  $|t| < \delta$ .*

**Lemma 3.2.** *Let  $\varphi$  be a  $C^1$  flow defined in a compact manifold  $M$  and  $\Lambda \subset M$  be a compact  $\varphi$ -invariant subset such that all singularities of  $\varphi$  in  $\Lambda$  are hyperbolic. Then, either  $\varphi|_\Lambda$  has no regular periodic orbits or*

$$\varepsilon_0(\varphi|_\Lambda) := \inf\{T > 0 : T \text{ is period of a regular periodic orbit of } \varphi|_\Lambda\} > 0.$$

*Proof.* Assume that  $\varphi|_\Lambda$  has regular periodic orbits. Since all singularities are hyperbolic and  $\Lambda$  is compact then these must be in finite number  $\sigma_1, \dots, \sigma_n$ . For all  $1 \leq i \leq n$  let  $V_i$  be a small neighborhood of  $\sigma_i$  given by Hartman-Grobman's Theorem (see [31], p. 68). Every small  $V_i$  which is neighborhood of a sink or source contains no regular periodic orbits. On other hand, if a periodic orbit intersects a neighborhood  $V_i$  associated with a hyperbolic singularity of saddle type then its period is bounded below by a uniform constant (which is inversely proportional to the largest eigenvalue of the unstable bundle among the hyperbolic saddles). It remains to prove that all periodic orbits in  $S = \Lambda \cap (M \setminus \bigcup_{i=1}^n V_i)$  have a period bounded away from zero. By construction  $S$  is a compact set without singularities. For every  $x \in S$  let  $\delta_x > 0$  and  $B_x = U_x^{\delta_x}$  be a tubular neighborhood associated to  $x$ . By compactness of  $S$  the open covering  $(B_x)_{x \in S}$  admits a finite cover  $(B_{x_j})_{j=1}^\kappa$ . It is now clear that any periodic orbit in  $S$  has period larger or equal to  $\min_{1 \leq j \leq \kappa} \delta_{x_j}$ . Thus  $\varepsilon_0(\varphi|_\Lambda) > 0$ , which finishes the proof of the lemma.  $\square$

**3.2. Expansiveness and local triviality at regular points.** In the next lemma, we prove the existence, uniqueness and continuity of a local reparametrization for an element in the centralizer of an expansive flow

*varphi*. In the case that  $\varphi$  has no regular period orbits on  $\Lambda$  set for simplicity  $\varepsilon_0(\varphi|_\Lambda) = +\infty$ .

**Lemma 3.3.** *Let  $\varphi$  a  $C^1$  flow on a compact manifold  $M$  and  $\Lambda \subset M$  be a compact  $\varphi$ -invariant subset such that the restriction  $\varphi|_\Lambda$  is expansive and all the singularities of  $\varphi$  in  $\Lambda$  are hyperbolic. If  $\psi = (\psi_s)_{s \in \mathbb{R}}$  belongs to  $\mathcal{Z}^1(\varphi|_\Lambda)$  then for any  $0 < \varepsilon < \varepsilon_0(\varphi|_\Lambda)/3$  there exists  $\mu > 0$  and a unique function  $z : [-\mu, \mu] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow (-\varepsilon, \varepsilon)$  such that  $\psi_s(x) = \varphi_{z(s, x)}(x)$  for any  $(s, x) \in [-\mu, \mu] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ . Moreover,*

- (i)  $z$  is continuous,
- (ii) if  $t, s, t + s \in [-\mu, \mu]$ ,  $z(t + s, x) = z(t, x) + z(s, \psi(t, x))$ .

*Proof.* By Lemma 3.2 we have that  $\varepsilon_0(\varphi|_\Lambda) > 0$ . Given  $0 < \varepsilon < \varepsilon_0(\varphi)/3$ , let  $\delta > 0$  be given by the expansiveness property (recall Definition 2.3). Since  $\Lambda$  is compact and  $\varphi$ -invariant then there exists  $\mu > 0$  such that  $\sup_{|s| \leq \mu} \{d(\text{Id}, \psi_s)\} < \delta$

and, consequently,

$$d(\varphi_t(x), \varphi_t(\psi_s(x))) = d(\varphi_t(\psi_0(x)), \varphi_t(\psi_s(x))) = d(\psi_0(\varphi_t(x)), \psi_s(\varphi_t(x))) < \delta$$

for every  $x \in \Lambda$ ,  $|s| \leq \mu$  and  $t \in \mathbb{R}$ . This implies (taking  $h(t) = t$  in Definition 2.3) that there exists  $t_0 \in \mathbb{R}$  such that  $\varphi_{t_0}(\psi_s(x)) = \varphi_{t_0+\eta}(x)$  for some  $\eta \in (-\varepsilon, \varepsilon)$ . Consequently  $\psi_s(x) = \varphi_\eta(x)$  belongs to the orbit of  $x$  relative to the flow  $(\varphi_t)_t$ .

This defines uniquely a map  $z : [-\mu, \mu] \times \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow (-\varepsilon, \varepsilon)$  such that  $\psi_s(x) = \varphi_{z(s,x)}(x)$  for any  $(s, x) \in [-\mu, \mu] \times \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ . In fact, if  $z_1, z_2 : [-\mu, \mu] \times \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow (-\varepsilon, \varepsilon)$  are such that  $\varphi_{z_1(s,x)}(x) = \psi_s(x) = \varphi_{z_2(s,x)}(x)$  then  $\varphi_{z_1(s,x)-z_2(s,x)}(x) = x$  with  $|z_1(s,x) - z_2(s,x)| \leq |z_1(s,x)| + |z_2(s,x)| < 2/3 \varepsilon_0(\varphi)$ . So, by definition of  $\varepsilon_0(\varphi)$  we conclude that  $z_1(s,x) = z_2(s,x)$ .

To prove (i) assume by contradiction that  $z$  is not continuous. Then there are  $\delta_0 > 0$ ,  $(s, x) \in [-\mu, \mu] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$  and a sequence  $(s_n, x_n)_{n \in \mathbb{N}}$  in  $[-\mu, \mu] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$  that converges to  $(s, x)$  and such that  $|z(s_n, x_n) - z(s, x)| \geq \delta_0$  for all  $n \in \mathbb{N}$ . This implies there exists  $\delta_1 > 0$  such that  $d(\varphi_{z(s_n, x_n)}(x), \varphi_{z(s, x)}(x)) \geq \delta_1$  for all  $n \in \mathbb{N}$ . On the other hand

$$\begin{aligned} \delta_1 &\leq d(\varphi_{z(s_n, x_n)}(x), \varphi_{z(s, x)}(x)) \\ &\leq d(\varphi_{z(s_n, x_n)}(x), \varphi_{z(s_n, x_n)}(x_n)) + d(\varphi_{z(s_n, x_n)}(x_n), \varphi_{z(s, x)}(x)) \\ &= d(\varphi_{z(s_n, x_n)}(x), \varphi_{z(s_n, x_n)}(x_n)) + d(\psi_{s_n}(x_n), \psi_s(x)). \end{aligned}$$

Using further that  $(z(s_n, x_n))_{n \in \mathbb{N}}$  is bounded then, up to consider a subsequence, we may assume without loss of generality that  $z(s_n, x_n) \rightarrow t_0$  as  $n \rightarrow \infty$ . In consequence, the right hand side above tends to zero by continuity of the flow  $\varphi$  and of the time  $t_0$ -map  $\varphi_{t_0}$ , contradicting the existence of  $\delta_1 > 0$ . This proves the continuity claimed in (i). To prove the property (ii), by the equality

$$\begin{aligned} \varphi_{z(t+s, x)}(x) &= \psi_{t+s}(x) = \psi_t(\psi_s(x)) \\ &= \varphi_{z(t, \psi_s(x))}(\psi_s(x)) \\ &= \varphi_{z(t, \psi_s(x))}(\varphi_{z(s, x)}(x)) \\ &= \varphi(z(t, \psi_s(x)) + z(s, x), x) \end{aligned}$$

and uniqueness of local reparametrization  $z$ , for  $0 < \varepsilon < \varepsilon_0(\varphi)/3$  we conclude that  $z(t+s, x) = z(t, \psi(s, x)) + z(s, x)$  for any  $t, s, t+s \in [-\mu, \mu]$  and every regular point  $x \in \Lambda$ .  $\square$

**3.3. Unique continuous extension for the local reparametrization.** In what follows we construct an extension of the continuous reparametrization described in Lemma 3.3 to  $\mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ . More precisely:

**Proposition 3.1.** *Let  $\varphi$  be a continuous flow in  $M$  and  $\psi$  be a continuous flow such that there exists  $\mu > 0$  and  $z : [-\mu, \mu] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow (-\varepsilon, \varepsilon)$  such that  $\psi(s, x) = \varphi(z(s, x), x)$  for any  $(s, x) \in [-\mu, \mu] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ , where  $0 < \varepsilon < \varepsilon_0(\varphi)/3$ . There exists a unique continuous function  $p : \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow \mathbb{R}$  which extends  $z$  and satisfies  $\psi(s, x) = \varphi(p(s, x), x)$  for any  $(s, x) \in \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ .*

*Proof.* We first prove the existence of the reparametrization. Take  $N \geq 1$  so that  $2^{-N} < \mu$ . Now, let  $z_1 : [1/2^N, 2/2^N] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow \mathbb{R}$  be the continuous

function given by

$$z_1(t, x) = z\left(t - \frac{1}{2^N}, x\right) + z\left(\frac{1}{2^N}, \psi\left(t - \frac{1}{2^N}, x\right)\right).$$

A simple computation shows that  $z_1(\frac{1}{2^N}, x) = z(\frac{1}{2^N}, x)$  for any point  $x \in \Lambda \setminus \text{Sing}(\varphi|_\Lambda)$ . This means that the functions coincide in the extreme point of the interval  $[0, 1/2^N]$ . Now we claim that  $\psi(t, x) = \varphi(z_1(t, x), x)$  for every  $(t, x) \in [1/2^N, 2/2^N] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ . Fix  $(t, x) \in [1/2^N, 2/2^N] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ . On the one hand, by Lemma 3.3,

$$0 = z\left(\frac{1}{2^N} - \frac{1}{2^N}, \psi(t, x)\right) = z\left(\frac{1}{2^N}, \psi\left(t - \frac{1}{2^N}, x\right)\right) + z\left(-\frac{1}{2^N}, \psi(t, x)\right)$$

and consequently,  $z(-1/2^N, \psi(t, x)) = -z(1/2^N, \psi(t - 1/2^N, x))$  for every  $(t, x) \in [1/2^N, 2/2^N] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ . On the other hand, since  $|z(\cdot)|$  is bounded above by  $\varepsilon_0(\varphi|_\Lambda)/3$ ,

$$\begin{aligned} & \varphi\left(-z\left(\frac{1}{2^N}, \psi\left(t - \frac{1}{2^N}, x\right)\right), \varphi(z_1(x, t), x)\right) \\ &= \varphi\left(-z\left(\frac{1}{2^N}, \psi\left(t - \frac{1}{2^N}, x\right)\right) + z_1(x, t), x\right) \\ &= \varphi\left(z\left(t - \frac{1}{2^N}, x\right), x\right) \\ &= \psi\left(t - \frac{1}{2^N}, x\right) \\ &= \psi\left(-\frac{1}{2^N}, \psi(t, x)\right) \\ &= \varphi\left(z\left(-\frac{1}{2^N}, \psi(t, x)\right), \psi(t, x)\right) \\ &= \varphi\left(-z\left(\frac{1}{2^N}, \psi\left(t - \frac{1}{2^N}, x\right)\right), \psi(t, x)\right) \end{aligned}$$

from which the claim follows. This proves our claim and so  $\psi$  is a local reparameterization of the flow  $\varphi$  on the interval  $[1/2^N, 2/2^N]$ . Inductively, for each integer  $k \geq 1$  let  $z_k : [\frac{k}{2^N}, \frac{k+1}{2^N}] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow \mathbb{R}$  be the continuous function given by

$$z_k(t, x) = z\left(t - \frac{k}{2^N}, x\right) + \sum_{i=1}^k z\left(\frac{1}{2^N}, \psi\left(t - \frac{i}{2^N}, x\right)\right).$$

For any  $x \in \Lambda \setminus \text{Sing}(\varphi)$  we observe that  $z_k(\frac{k+1}{2^N}, x) = z_{k+1}(\frac{k+1}{2^N}, x)$  because

$$\begin{aligned} z_k\left(\frac{k+1}{2^N}, x\right) &= z\left(\frac{1}{2^N}, x\right) + \sum_{i=1}^k z\left(\frac{1}{2^N}, \psi\left(\frac{k+1-i}{2^N}, x\right)\right) \\ &= z\left(\frac{1}{2^N}, x\right) + \sum_{j=1}^k z\left(\frac{1}{2^N}, \psi\left(\frac{j}{2^N}, x\right)\right) = \sum_{j=0}^k z\left(\frac{1}{2^N}, \psi\left(\frac{j}{2^N}, x\right)\right) \\ &= z_{k+1}\left(\frac{k+1}{2^N}, x\right). \end{aligned}$$

In addition,  $z_k$  satisfies the recursive expression

$$z_k(t, x) = z_{k-1}(t - 1/2^N, x) + z(1/2^N, \psi(t - 1/2^N, x)). \quad (3.1)$$

Indeed, simple computations yield

$$\begin{aligned}
z_k(t, x) &= z\left(t - \frac{k}{2^N}, x\right) + \sum_{i=1}^k z\left(\frac{1}{2^N}, \psi\left(t - \frac{i}{2^N}, x\right)\right) \\
&= z\left(\left(t - \frac{1}{2^N}\right) - \frac{k-1}{2^N}, x\right) + \sum_{i=1}^k z\left(\frac{1}{2^N}, \psi\left(\left(t - \frac{1}{2^N}\right) - \frac{i+1}{2^N}, x\right)\right) \\
&= z\left(\left(t - \frac{1}{2^N}\right) - \frac{k-1}{2^N}, x\right) + \sum_{i=1}^{k-1} z\left(\frac{1}{2^N}, \psi\left(\left(t - \frac{1}{2^N}\right) - \frac{i}{2^N}, x\right)\right) \\
&\quad + z\left(\frac{1}{2^N}, \psi\left(t - \frac{1}{2^N}, x\right)\right)
\end{aligned}$$

and proves the equality in (3.1). We need the following:

**Claim:** For every  $x \in \Lambda \setminus \text{Sing}(\varphi|_\Lambda)$  and  $t \in \left[\frac{k}{2^N}, \frac{k+1}{2^N}\right]$  the function  $z_k$  satisfies  $\psi(t, x) = \varphi(z_k(t, x), x)$ .

*Proof of the claim.* The claim for  $k = 1$  is trivial. By induction, assume that the affirmation is true for  $k - 1$ . Then, using (3.1), we obtain that

$$\begin{aligned}
\varphi\left(-z\left(\frac{1}{2^N}, \psi\left(t - \frac{1}{2^N}, x\right)\right), \varphi(z_k(t, x), x)\right) &= \varphi\left(-z\left(\frac{1}{2^N}, \psi\left(t - \frac{1}{2^N}, x\right)\right) + z_k(t, x), x\right) \\
&= \varphi\left(z_{k-1}\left(t - \frac{1}{2^N}, x\right), x\right) \\
&= \psi\left(t - \frac{1}{2^N}, x\right) = \psi\left(-\frac{1}{2^N}, \psi(t, x)\right) \\
&= \varphi\left(z\left(-\frac{1}{2^N}, \psi(t, x)\right), \psi(t, x)\right) \\
&= \varphi\left(-z\left(\frac{1}{2^N}, \psi\left(t - \frac{1}{2^N}, x\right)\right), \psi(t, x)\right).
\end{aligned}$$

Since the time- $s$  map  $\varphi_s$  is a diffeomorphism for all  $s \in \mathbb{R}$ , we conclude that  $\varphi(t, x) = \varphi(z_k(t, x), x)$  for every  $x \in \Lambda \setminus \text{Sing}(\varphi|_\Lambda)$  and  $t \in \left[\frac{k}{2^N}, \frac{k+1}{2^N}\right]$ , which proves the claim.  $\square$

Clearly, a completely similar argument as above is enough to extend  $z(t, x)$  for all negative  $t$ . Just consider the continuous function  $\bar{z}_1 : [-2/2^N, -1/2^N] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow \mathbb{R}$  given by

$$\bar{z}_1(t, x) = z\left(t + \frac{1}{2^N}, x\right) + z\left(-\frac{1}{2^N}, \varphi\left(t + \frac{1}{2^N}, x\right)\right).$$

Computations similar to the ones above yield that for any  $x \in \Lambda \setminus \text{Sing}(\varphi|_\Lambda)$ , the function  $\bar{z}_1(\cdot, x)$  satisfies  $\bar{z}_1(-1/2^N, x) = z(-1/2^N, x)$  and  $\varphi(t, x) = \varphi(\bar{z}_1(t, x), x)$ . Then, for each positive integer  $k$ , take  $\bar{z}_k : [-(k+1)/2^N, -k/2^N] \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow \mathbb{R}$  given by

$$\bar{z}_k(t, x) = z\left(t + \frac{k}{2^N}, x\right) + \sum_{i=1}^k z\left(-\frac{1}{2^N}, \varphi\left(t + \frac{i}{2^N}, x\right)\right),$$

which satisfies  $\varphi(t, x) = \varphi(\bar{z}_k(t, x), x)$  for every  $t \in [-(k+1)/2^N, -k/2^N]$  and  $x \in \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ . Altogether we get a well-defined continuous function  $p : \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow \mathbb{R}$  given by

$$p(t, x) = \begin{cases} z(t, x), & \text{if } t \in [-1/2^N, 1/2^N] \\ z_k(t, x), & \text{if } t \in [k/2^N, (k+1)/2^N] \\ \bar{z}_k(t, x), & \text{if } t \in [-(k+1)/2^N, -k/2^N] \end{cases}, k \in \mathbb{N}.$$

By construction, for every  $\psi \in \mathcal{Z}^\infty(\varphi)$  there exists  $p$  is continuous and such that  $\psi(t, x) = \varphi(p(t, x), x)$  for every  $(t, x) \in \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ . This concludes the proof of the existence of the reparametrization.

In the remaining of the proof of the proposition we are left to prove the uniqueness of the reparametrization among non-singular points. For this, suppose there are  $p_1, p_2 : \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow \mathbb{R}$ , continuous extensions of  $z$  such that  $\varphi(p_1(t, x), x) = \psi(t, x) = \varphi(p_2(t, x), x)$  for all  $(t, x) \in \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ . Fix  $x \in \Lambda \setminus \text{Sing}(\varphi|_\Lambda)$  and consider the continuous function  $\alpha_x(t) = p_1(t, x) - p_2(t, x)$ . Observe that  $\alpha_x^{-1}(0) \supset [-\mu, \mu]$  (thus  $\alpha_x^{-1}(0)$  is non empty) and  $\alpha_x^{-1}(0)$  is closed by continuity of  $\alpha_x$ .

Assume, by contradiction, that  $\alpha_x^{-1}(0) \neq \mathbb{R}$ . Then, as  $\alpha_x$  is continuous, there exists  $t_0 = \max\{t > 0 : [0, t] \subset \alpha_x^{-1}(0)\} \geq \mu$  or  $\min\{t < 0 : [t, 0] \subset \alpha_x^{-1}(0)\} \leq -\mu$ . We assume the first case holds (the second is completely analogous). By continuity of the reparametrizations  $p_1(t_0, x) = p_2(t_0, x)$ . Moreover, if  $t \in [-\mu, \mu]$ , then  $\varphi(p_i(t + t_0, x), x) = \psi(t + t_0, x) = \psi(t, \varphi(t_0, x))$  for  $i \in \{1, 2\}$ . By Lemma 3.3 we know the existence of a unique function  $z : [-\mu, \mu] \times \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda)) \rightarrow (-\varepsilon, \varepsilon)$  such that  $\psi(s, x) = \varphi(z(s, x), x)$  for any  $(s, x) \in [-\mu, \mu] \times \mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$ . In particular,

$$\begin{aligned} \psi(t, \varphi(t_0, x)) &= \varphi(z(t, \varphi(t_0, x)), \varphi(t_0, x)) \\ &= \varphi(z(t, \varphi(t_0, x)), \varphi(p(t_0, x), x)) \\ &= \varphi(z(t, \varphi(t_0, x)) + p(t_0, x), x), \end{aligned}$$

which contradicts the maximality of  $t_0$ . Consequently, we conclude that  $\alpha_x^{-1}(0) = \mathbb{R}$  and that the reparametrizations  $p_1$  and  $p_2$  do coincide. This completes the proof of Proposition 3.1.  $\square$

**3.4. Invariance of reparametrizations along regular orbits.** In what follows we prove that the unique reparameterization obtained in Proposition 3.1 is invariant along orbits of regular points.

**Lemma 3.4.** *If  $p$  is the reparametrization given in Proposition 3.1 then  $p(t, x) = p(t, \varphi(s, x))$  for every  $t \in \mathbb{R}$  and any  $x \in \Lambda \setminus \text{Sing}(\varphi|_\Lambda)$ . Moreover, there exists a unique continuous function  $A : \Lambda \setminus \text{Sing}(\varphi|_\Lambda) \rightarrow \mathbb{R}$  so that  $p(t, x) = A(x)t$  for every  $t \in \mathbb{R}$  and  $x \in \Lambda \setminus \text{Sing}(\varphi|_\Lambda)$ .*

*Proof.* Initially we observe that, since  $\psi$  commutes with  $\varphi$ ,

$$\begin{aligned} \varphi(s + p(t, x), x) &= \varphi(s, \varphi(p(t, x), x)) = \varphi(s, \psi(t, x)) \\ &= \psi(t, \varphi(s, x)) = \varphi(p(t, \varphi(s, x)), \varphi(s, x)) \\ &= \varphi(p(t, \varphi(s, x)) + s, x). \end{aligned}$$

Therefore, for  $\mu$  sufficiently small and  $t \in [-\mu, \mu]$  this equality implies that  $p(t, x) = p(t, \varphi(s, x))$  for every  $s \in \mathbb{R}$ . From the construction and uniqueness of function  $p$

together with the recursive expression (3.1) it follows that  $p(t, x) = p(t, \varphi(s, x))$  for all  $t, s \in \mathbb{R}$ . Using that  $p(t, x) = p(t, \varphi(s, x))$  for every  $s \in \mathbb{R}$  together with Proposition 3.1,

$$\begin{aligned}\varphi(p(t+s, x), x) &= \psi(t+s, x) = \psi(t, \psi(s, x)) \\ &= \varphi(p(t, \psi(s, x)), \psi(s, x)) \\ &= \varphi(p(t, \varphi(p(s, x), x)), \varphi(p(s, x), x)) \\ &= \varphi(p(t, \varphi(p(s, x), x)) + p(s, x), x) \\ &= \varphi(p(t, x) + p(s, x), x)\end{aligned}$$

for all  $t \in \mathbb{R}$ . The uniqueness of  $p$  implies that  $p(t+s, x) = p(t, x) + p(s, x)$  for all  $t, s$ . Since  $p(\cdot, x)$  is continuous then it is linear. Hence there exists a continuous map  $A : M \rightarrow \mathbb{R}$  so that  $p(t, x) = A(x)t$  for all  $x \in \Lambda \setminus \text{Sing}(\varphi|_\Lambda)$ . This finishes the proof of the lemma.  $\square$

**3.5. Extension of the reparametrization to singular points.** Under our assumptions on the singularities the reparametrization obtained Proposition 3.1 we will prove to extend continuously to the singular points. This will complete the proof of Theorem A. For this, initially we deduce the following version of Kopell's theorem ([21], Theorem 6) for linear contractions. For any complex number  $\lambda \in \mathbb{C}$  let  $\text{Re}(\lambda)$  denote its real part.

**Lemma 3.5.** *Given  $B \in GL(n, \mathbb{R})$  let  $\varphi = (e^{t \cdot B})_{t \in \mathbb{R}}$  be such that 0 is a sink and assume that it has non-ressonant eigenvalues. If  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $B$  and  $m$  is the least positive integer such that*

$$m \cdot \left( \max_{1 \leq i \leq n} \text{Re}(\lambda_i) \right) < \min_{1 \leq i \leq n} \text{Re}(\lambda_i) \quad (3.2)$$

*then  $\mathcal{Z}^m(\varphi)$  is the set of linear flows  $(e^{s \cdot C})_{s \in \mathbb{R}}$  where  $C \in GL(n, \mathbb{R})$  is such that  $B \cdot C = C \cdot B$ .*

*Proof.* As 0 is a sink for  $B$  then  $e^B$  is a linear contraction and, the fact that the eigenvalues of  $B$  satisfy condition (3.2) on the eigenvalues of  $B$  implies that the non-ressonance conditions in [21, Theorem 6]. Since 0 is a sink then all eigenvalues  $\lambda_1, \dots, \lambda_n$  of  $B$  have real negative part and  $|e^{\lambda_i}| < 1$  for all  $1 \leq i \leq n$ . Thus, if  $m$  is given by (3.2) above, the result of Kopell implies that the  $\mathcal{C}^m$ -centralizer of the linear automorphism  $e^B$  acting on  $\mathbb{R}^n$  is constituted exclusively by linear maps. So, any element of  $\mathcal{Z}^m(\varphi)$  is a flow  $\psi = (\psi_s)_{s \in \mathbb{R}}$  such that  $\psi_s$  is a linear map for all  $s \in \mathbb{R}$  (since all of these should commute with the time-1 map  $e^B$ ).

In what follows we show that any flow of linear maps  $\psi \in \mathcal{Z}^m(\varphi)$  is of exponential type. We claim that  $\psi_s = e^{sC}$  for every  $s \in \mathbb{R}$ , where

$$C(x) := \lim_{s \rightarrow 0} \frac{\psi_s(x) - x}{s} = \frac{\partial \psi_s}{\partial s} \Big|_{s=0} (x)$$

for  $x \in \mathbb{R}^n$ . Let  $T > 0$  be fixed. Since the maps  $t \rightarrow \|e^{tC}\|$  and  $s \rightarrow \|\psi_s\|$  are continuous there exists a constant  $K > 0$  (depending on  $T$ ) such that  $\|\psi_s\| \cdot \|e^{tC}\| \leq K$  for all  $0 \leq t, s \leq T$ . Given  $\varepsilon > 0$ , as  $C = \lim_{s \rightarrow 0} \frac{e^{sC} - Id}{s}$ , one can choose  $0 < \delta \leq T$  such that

$$\left\| \frac{\psi_h - e^{hC}}{h} \right\| < \frac{\varepsilon}{KT} \quad \text{for all } 0 \leq h \leq \delta.$$



Now pick  $n \in \mathbb{N}$  such that  $\frac{T}{n} < \delta$ . Then, using the triangular inequality, for any  $t \in [0, T]$

$$\begin{aligned} \|\psi_t - e^{tC}\| &= \|\psi_{n \cdot \frac{t}{n}} - e^{n \cdot \frac{t}{n} C}\| \\ &\leq \sum_{k=0}^{n-1} \|\psi_{(n-k) \cdot \frac{t}{n}} \circ e^{k \frac{t}{n} C} - \psi_{(n-k-1) \cdot \frac{t}{n}} \circ e^{(k+1) \frac{t}{n} C}\| \\ &\leq \sum_{k=0}^{n-1} \|\psi_{(n-k-1) \cdot \frac{t}{n}}\| \cdot \|\psi_{\frac{t}{n}} - e^{\frac{t}{n} C}\| \cdot \|e^{k \frac{t}{n} C}\| \\ &\leq Kn \cdot \frac{\varepsilon}{KT} \cdot \frac{t}{n} < \varepsilon. \end{aligned}$$

Since  $\varepsilon$  was chosen arbitrary the later proves that  $\psi_t = e^{tC}$  for all  $t \in [0, T]$ . The group property implies that the equality holds for all  $t \in \mathbb{R}$ . Finally, such flows commute if and only if  $BC = CB$ . This completes the proof of the lemma.  $\square$

Next, we use Lemma 3.5 to prove that the reparametrization can be continuously extended to the singularities  $\sigma_i$ . Indeed, we will prove that the restriction of the local reparametrizations to the stable and unstable manifolds  $W^s(\sigma_i) \setminus \sigma_i$  and  $W^u(\sigma_i) \setminus \sigma_i$  of a non-ressonant hyperbolic singularity  $\sigma_i$  of a  $\mathcal{C}^\infty$ -flow  $\varphi$  are necessarily constant. This, together with the fact that

$$\mathcal{Z}^\infty(\varphi) \subset \mathcal{Z}^m(\varphi) \subset \mathcal{Z}^1(\varphi)$$

for every  $m \in \mathbb{N}$  and every  $\mathcal{C}^\infty$ -flow  $\varphi$  be a key step in the proof of the theorem.

**Lemma 3.6.** *Let  $\varphi : \mathbb{R} \times \Lambda \rightarrow \Lambda$  a  $\mathcal{C}^\infty$ -expansive flow defined on a compact Riemannian manifold  $M$ . Suppose that the singularities  $\sigma_1, \dots, \sigma_k$  of  $\varphi$  are hyperbolic, set  $B_i = \frac{d}{dt}\varphi(t, \sigma_i)|_{t=0}$  and let  $T_{\sigma_i}M = E_i^s \oplus E_i^u$  be the hyperbolic splitting,  $1 \leq i \leq k$ . If the eigenvalues of  $B_i|_{E_i^*}$  are distinct and non resonant for every  $1 \leq i \leq k$  and  $* \in \{s, u\}$  then the continuous function  $A(\cdot)$  given by Lemma 3.4 admits a continuous extension to  $\Lambda$ .*

*Proof.* Since the singularities of  $\varphi$  are hyperbolic then these are isolated and, for that reason, it is enough to extend the function  $A(\cdot)$  to each singularity recursively. We subdivide the proof in two cases, corresponding to the case where the singularities are either sinks/sources or saddles.

*Case 1:*  $\sigma_i$  is a sink or a source.

Assume without loss of generality that  $\sigma_i$  is a sink. Indeed, in the case that  $\sigma_i$  is a source the proof is completely analogous just by considering the time reversed flow  $(\varphi_{-t})_{t \in \mathbb{R}}$ . Since the eigenvalues of  $B_i = \frac{d}{dt}\varphi(t, \sigma_i)|_{t=0}$  are non-ressonant, by Sternberg linearization theorem (see [40]) there exists a neighborhood  $W_i$  of  $\sigma_i$  such that the flow  $(\varphi_t)_t$  is  $\mathcal{C}^\infty$ -linearizable in  $W_i$ : there exists a  $\mathcal{C}^\infty$ -chart  $\zeta_i$  on an open set in  $\mathbb{R}^{\dim M}$  so that  $\eta_t := \zeta_i \circ \varphi_t \circ \zeta_i^{-1}$ ,  $t \in \mathbb{R}$ , defines a linear flow on  $W_i$ . This conjugation  $\zeta_i$  induces a natural isomorphism between  $\mathcal{Z}^\infty((\varphi_t)_{t \in \mathbb{R}})$  and  $\mathcal{Z}^\infty((\eta_s)_{s \in \mathbb{R}})$  in the sense that  $(\psi_s)_{s \in \mathbb{R}} \in \mathcal{Z}^\infty((\varphi_t)_{t \in \mathbb{R}})$  if and only if  $(h \circ \psi_s \circ h^{-1})_{s \in \mathbb{R}} \in \mathcal{Z}^\infty((\eta_t)_{t \in \mathbb{R}})$ .

We use this fact to determine the centralizer of  $\varphi$  in a neighborhood of the singularities  $\sigma_i$ . On the one hand, Lemma 3.5 implies that the centralizer  $\mathcal{Z}^\infty((e^{sB_i})_{s \in \mathbb{R}})$  is formed by the linear flows  $(e^{sC})_{s \in \mathbb{R}}$  where the linear map  $C$  satisfies  $B_i \cdot C = C \cdot B_i$ . On the other hand, it follows from Lemmas 3.3 and 3.4

and Proposition 3.1 that any  $\psi \in \mathcal{Z}^\infty(\varphi)$  is reparametrization of  $\varphi$ , meaning that there exists continuous function  $A : W_i \setminus \{\sigma_i\} \rightarrow \mathbb{R}$  so that  $\psi_t(x) = \varphi_{A(x) \cdot t}(x)$  for all  $t \in \mathbb{R}$ . Thus we are interested in determining which linear flows  $(e^{sC})_{s \in \mathbb{R}}$  preserve the orbits of the linear flow  $(e^{tB_i})_{t \in \mathbb{R}}$ . We claim that all such flows are of the form  $(e^{sC})_{s \in \mathbb{R}}$  with  $C = cB_i$ , for some  $c \in \mathbb{R}$ . In fact, if  $C \neq cB_i$  for all  $c \in \mathbb{R}$  then there would exist  $x \in \mathbb{R}^n$  such that the vectors  $\{B_i(x), C(x)\}$  are linearly independent and, consequently, the orbits of  $x$  by the two flows are transversal at  $x$ . This would contradict the fact that  $(e^{sC}x)_{s \in \mathbb{R}}$  is a reparametrization of  $(e^{tB_i}x)_{t \in \mathbb{R}}$  and proves the claim. Finally we conclude via the conjugation given by Sternberg linearization theorem that, in the linearizing coordinates, the function  $A(x)$  given by Lemma 3.4 is constant in  $W^s(\sigma_i) \setminus \{\sigma_i\}$ , hence it admits a continuous extension to  $\sigma_i$ .

*Case 2:  $\sigma_i$  is a saddle.*

Let  $W_i$  be a neighborhood of  $\sigma_i$  given by Hartman-Grobman's theorem: the flow  $\varphi$  is topologically conjugate to the hyperbolic linear flow  $(e^{tB_i})_{t \in \mathbb{R}}$  on  $W_i$ , where  $B_i = \frac{d}{dt}\varphi(t, \sigma_i)|_{t=0}$ . Choose a suitable base of  $\mathbb{R}^d$  so that  $B_i = \text{diag}\{B_{1,i}, B_{2,i}\}$  and  $B_{1,i} \in B_{2,i}^{-1}$  are contractions. Moreover, since the eigenvalues of  $B$  are non-ressonant, we may reduce  $W_i$  if necessary to guarantee that the flow  $\varphi$  restricted to the open neighborhood  $S_i := W_i \cap W^s(\sigma)$  of  $p$  in  $W^s(\sigma_i)$  (resp. open neighborhood  $U_i := W_i \cap W^u(\sigma)$  of  $p$  in  $W^u(\sigma_i)$ ) is  $C^\infty$ -conjugate to the linear contraction  $(e^{tB_{1,i}})_{t \in \mathbb{R}}$  (resp. to the linear expansion  $(e^{tB_{2,i}})_{t \in \mathbb{R}}$ ). Using Case 1 to deal, independently, with both linear flows we deduce that the reparametrization  $A(x)$  given by Lemma 3.4 is constant along both invariant manifolds  $W^s(\sigma_i) \setminus \{\sigma_i\}$  and  $W^u(\sigma_i) \setminus \{\sigma_i\}$ . To conclude that  $A(x)$  extends continuously to  $\sigma_i$  it is enough to show that, in the linearizing coordinates, there exists  $c \in \mathbb{R}$  so that  $A(x) = c$  is constant for all  $x \in [W^s(\sigma_i) \cup W^u(\sigma_i)] \setminus \{\sigma_i\}$ . For this, consider a compact cross-section  $\Sigma$  that is transversal to  $W^s(\sigma_i)$ , a compact section  $\Sigma'$  that it is transversal to  $W^u(\sigma_i)$ , a point  $x \in \Sigma \cap W^s(\sigma_i)$  and a sequence  $(x_n)$  of regular points in  $\Sigma \setminus (W^s(\sigma_i) \cup W^u(\sigma_i))$  such that  $x_n \rightarrow x$  when  $n \rightarrow \infty$ . For all  $n \geq 1$  large there exists a sequence  $(t_n)$  in  $\mathbb{R}$  such that  $\varphi(A(x_n)t_n, x_n) \in \Sigma'$ . By compactness of  $\Sigma'$  there exists a convergent subsequence  $(\varphi(A(x_{n_k})t_{n_k}, x_{n_k}))_{k \geq 1}$ . Denoting by  $x'$  the limit of the subsequence, the continuity of  $A$  in  $\mathbb{R} \times (\Lambda \setminus \text{Sing}(\varphi|_\Lambda))$  and its invariance along the orbits (cf. Proposition 3.1 and Lemma 3.4) implies that

$$\begin{aligned} A(x) &= \lim_{k \rightarrow \infty} A(x_{n_k}) = \lim_{k \rightarrow \infty} A(\varphi(A(x_{n_k})t_{n_k}, x_{n_k})) \\ &= A(\lim_{k \rightarrow \infty} \varphi(A(x_{n_k})t_{n_k}, x_{n_k})) = A(x'). \end{aligned}$$

Thus  $A$  extends continuously and uniquely to a reparametrization on  $M$  so that  $\psi_t(x) = \varphi_{A(x)t}(x)$  for all  $(t, x) \in \mathbb{R} \times M$ ,  $\psi \in \mathcal{Z}^\infty(\varphi)$ . This completes the proof of the lemma.  $\square$

To complete the proof of Theorem A it remains to prove that the reparametrization  $A$  is  $C^\infty$ -smooth. First assume  $x$  is a regular point for  $(\varphi_t)_t$  and  $W \subset M$  be a small open neighborhood of  $x$ . Denote by  $X$  the vector field associated to  $(\varphi_t)_t$ . Up to consider a change of coordinates by a chart, we will assume without loss of generality that  $W \subset \mathbb{R}^{\dim(M)}$  and that both  $\varphi = (\varphi_t)_t$  and  $\psi \in \mathcal{Z}^\infty(\varphi)$  are (locally) flows in  $\mathbb{R}^{\dim(M)}$ .

Fix  $t \neq 0$  and consider the  $C^\infty$ -function  $F(c, x) = \varphi_{ct}(z) - \psi_t(z)$  for  $c \in \mathbb{R}$  and  $z \in W$ . By construction,  $F(A(z), z) = 0$  for every  $z \in W$  (since the later is

equivalent to  $\psi_t(z) = \varphi_{A(x)t}(z)$ ). Since the partial derivative  $\frac{\partial F}{\partial c} \big|_{c_0} = X(\varphi_{c_0 t}(x)) \neq 0$  (because  $x$  is a regular point) then it follows from the implicit function theorem [34] that  $A(\cdot)$  has the same regularity of  $F$ . In other words,  $A$  is a  $C^\infty$ -function in  $W$ . It remains to check that  $A$  is  $C^\infty$  at the singularities. If  $\sigma$  is a hyperbolic singular point it can be a sink (or source) or a saddle. If  $\sigma$  is a sink (resp. source), since the reparametrization is constant in  $W^s(\sigma)$  (resp.  $W^u(\sigma)$ ) then  $A$  is constant in a neighborhood of  $x$  and it is trivially  $C^\infty$ . Finally, if  $\sigma$  is a saddle, the reparametrization  $A$  is constant in  $W^s(\sigma) \cup W^u(\sigma)$  and constant along orbits of the flow. Hence, the  $C^\infty$  regularity at a neighborhood of  $\sigma$  follows (similarly as done in the proof of Lemma 3.6) by establishing the continuity of all its derivatives on points forming a global cross-section to the flow on the local neighborhood of  $\sigma$ . This proves that  $A$  is  $C^\infty$  and completes the proof of Theorem A.

**3.6. Proof of Corollary A.** Let  $\varphi$  be a  $C^\infty$ -expansive flow on a compact and connected Riemannian manifold  $M$  whose singularities are hyperbolic and non-ressonant and suppose without loss of generality that  $\Lambda = \bigcup_{\sigma \in \text{Sing}(\varphi)} W^s(\sigma)$  (the other case is completely similar). If  $\psi \in \mathcal{Z}^\infty(\varphi)$ , Theorem A guarantees that there exists  $A : M \rightarrow \mathbb{R}$  such that  $\psi_t(x) = \varphi_{A(x)t}(x)$ . Moreover, the arguments used in the proof of Lemma 3.6 yield that this reparametrization  $A$  is constant in each  $W^s(\sigma_i)$ . Since there are finitely many singularities the image of  $A$  is constituted only by a finite number of elements. Now, using that  $M$  is connected and that  $A$  is continuous we conclude that  $A$  is constant. Consequently, there exist  $c \in \mathbb{R}$  so that  $\psi_t(x) = \varphi_{ct}(x)$ , for every  $t \in \mathbb{R}$  and  $x \in M$ . This proves the corollary.  $\square$

#### 4. EXPANSIVE $\mathbb{R}^d$ -ACTIONS AND SUSPENSIONS

In what follows we provide a characterization of expansive  $\mathbb{R}^d$ -actions obtained as suspensions of  $\mathbb{Z}^d$ -actions. Expansive subdynamics of  $\mathbb{Z}^d$ -actions on compact metric spaces have been considered e.g. in [10, 13]. Here we deal with expansive  $\mathbb{R}^d$ -actions and this first result is an extension of [11, Theorem 6], where Bowen and Walters proved that a continuous  $\mathbb{Z}$ -action is expansive if and only if its suspension flow is C-expansive. We first recall the notion of suspension action.

**4.1. Suspension of  $\mathbb{Z}^d$ -actions.** We first recall the notion of an  $\mathbb{R}^d$ -action that is obtained as *suspension of a  $\mathbb{Z}^d$ -action*. Let  $(e_i)_{i=1}^d$  be the canonical base on  $\mathbb{R}^d$ . Given a  $\mathbb{Z}^d$ -action,  $\varphi : \mathbb{Z}^d \times M \rightarrow M$  ( $d \geq 2$ ) we construct  $\mathbb{R}^d$ -actions that are suspensions of  $\varphi$ . Given a continuous roof function  $R : M \rightarrow (0, \infty)^d$  with  $R = (R_1, \dots, R_d)$  consider the set  $M_R = (M \times \mathbb{R}_+^d) / \sim_R$  where  $\sim_R$  is the equivalence relation

$(x, a_1, \dots, a_{i-1}, R_i(x), a_{i+1}, \dots, a_d) \sim_R (f_i(x), a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_d)$  for every  $x \in M$ ,  $0 \leq a_j \leq R_j(x)$ , where  $f_i(x) = \varphi(e_i, x)$ . Observe that  $\varphi((n_1, \dots, n_d), x) = f_1^{n_1} \circ \dots \circ f_d^{n_d}(x)$  for every integers  $n_i$  and  $x \in M$ . The suspension  $\mathbb{R}^d$ -action of  $\varphi : \mathbb{Z}^d \times M \rightarrow M$  with the roof function  $R$  is the action  $\Phi : \mathbb{R}^d \times M_R \rightarrow M_R$  is defined by

$$\begin{aligned} \Phi((t_1, \dots, t_d), (x, a_1, \dots, a_d)) &= \Phi(t_1 e_1 + \dots + t_d e_d, (x, a_1, \dots, a_d)) \\ &= \Phi(t_1 e_1, \Phi(t_2 e_2, \dots \Phi(t_d e_d, (x, a_1, \dots, a_d)))) \end{aligned}$$

where

$$\Phi(te_i, (x, a_1, \dots, a_d)) = (f_i^n(x), a_1, \dots, a_{i-1}, a_i + t - \sum_{j=0}^{n-1} R_i(f_i^j(x)), a_{i+1}, \dots, a_d)$$

and  $n \in \mathbb{Z}$  is uniquely determined by  $\sum_{j=0}^{n-1} R_i(f_i^j(x)) \leq a_i + t < \sum_{j=0}^n R_i(f_i^j(x))$ , for every  $x \in M$ ,  $1 \leq i \leq d$  and  $0 \leq a_i \leq R_i(x)$ . In this way, any  $\mathbb{Z}^d$ -action on a compact  $n$ -dimensional manifold  $M$  determines a  $\mathbb{R}^d$ -action  $\Phi$  on a  $n + d$ -dimensional compact manifold  $M_R$ .

The space  $M_R$  is metrizable and we exhibit a metric  $d$  that is compatible with the natural topology on  $M_R$  and is the analogous of the Bowen-Walters metric for flows. For the purposes of Theorem C it is enough to consider the roof function  $R$  constant to one and the corresponding space  $M_1$ . Let  $\rho$  denote the metric on  $M$ . Given  $M \times \{(t_1, \dots, t_d)\} \subset M_1$  and  $\sigma = (\sigma_1, \dots, \sigma_d) \in \{0, 1\}^d$ , consider the ‘horizontal’ distance  $\rho_h$  on  $M \times \{(t_1, \dots, t_d)\}$  defined by

$$\begin{aligned} \rho_h((x, t_1, \dots, t_d), (y, t_1, \dots, t_d)) \\ = \sum_{\sigma \in \{0, 1\}^d} \left\{ \prod_{i=1}^d [\sigma_i \cdot t_i + (1 - \sigma_i) \cdot (1 - t_i)] \right\} \rho(f_1^{\sigma_1} \circ \dots \circ f_d^{\sigma_d}(x), f_1^{\sigma_1} \circ \dots \circ f_d^{\sigma_d}(y)). \end{aligned}$$

It is not hard to show that

$$\sum_{\sigma \in \{0, 1\}^d} \left\{ \prod_{i=1}^d [\sigma_i \cdot t_i + (1 - \sigma_i) \cdot (1 - t_i)] \right\} = 1$$

and, consequently, defined in this way,  $\rho_h((x_1, t_1, \dots, t_d), (x_2, t_1, \dots, t_d))$  consists of the convex combination of the distances between the images of the points  $x, y$  and their iterates by maps of the form  $f_1^{\sigma_1} \circ \dots \circ f_d^{\sigma_d}$ . Second, in the particular case that  $d = 1$ ,  $\rho_h((x, t), (y, t)) = (1 - t)\rho(x, y) + t\rho(f(x), f(y))$  coincides with the metric introduced in [11] for suspension flows. Given any two points  $(x, t_1, \dots, t_d), (y, s_1, \dots, s_d) \in M_1$  consider the space of all the finite (admissible) sequences  $\omega_1 = (x_1, t_1, \dots, t_d), \dots, \omega_n = (x_n, s_1, \dots, s_d)$  such that  $x_1 = x, x_n = y$  and, for each  $1 \leq i \leq n - 1$ , either

- (1)  $\omega_i, \omega_{i+1} \in M \times \{(t_1, \dots, t_d)\}$  for some  $(t_1, \dots, t_d) \in [0, 1]^d$ , in which case we set  $\tilde{d}(\omega_i, \omega_{i+1}) = \rho_h(\omega_i, \omega_{i+1})$ ; or
- (2)  $\omega_i, \omega_{i+1}$  belong to the same orbit by the action  $\Phi$ , and we define  $\tilde{d}(\omega_i, \omega_{i+1}) := \inf\{\|v\| : \Phi_v \omega_i, \omega_{i+1}\}$  as the ‘vertical distance’ between  $\omega_i$  and  $\omega_{i+1}$  in  $M_1$ .

Finally, consider the metric in  $M_1$  given by

$$d((x, t_1, \dots, t_d), (y, s_1, \dots, s_d)) = \inf \sum_{i=1}^{n-1} \tilde{d}(\omega_i, \omega_{i+1}),$$

where the infimum is taken over the space of previously defined admissible sequences between  $(x, t_1, \dots, t_d)$  and  $(y, s_1, \dots, s_d)$ .

**4.2. Characterization of expansive  $\mathbb{R}^d$ -actions that are suspensions.** This section is devoted to the proof of the following characterization.

**Theorem C.** *Let  $M$  be a compact Riemannian manifold and  $1 : M \rightarrow (0, \infty)^d$  be the roof function constant to one. A continuous  $\mathbb{Z}^d$ -action  $\varphi : \mathbb{Z}^d \times M \rightarrow M$  is expansive if and only if its suspension  $\mathbb{R}^d$ -action  $\Phi : \mathbb{R}^d \times M_1 \rightarrow M_1$  is expansive.*

*Proof.* Suppose that the action  $(\Phi_v)_{v \in \mathbb{R}^d}$  is expansive (cf. Definition 2.4). So, given  $0 < \varepsilon < \frac{1}{2}$  let  $\delta > 0$  be so that if  $x, y \in M$  satisfy  $d(\Phi_v(x), \Phi_{h(v)}(y)) < \delta$  for every  $v \in \mathbb{R}^d$  with respect to a continuous function  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$  so that  $h(0) = 0$ , then  $y = \Phi_{v_0}(x)$  for some  $\|v_0\| < \varepsilon$ . We claim that the  $\mathbb{Z}^d$ -action  $\varphi$  is expansive. Assume that  $x_1, x_2 \in M$  are such that  $\rho(\varphi_{(n_1, \dots, n_d)}(x_1), \varphi_{(n_1, \dots, n_d)}(x_2)) < \delta$  for all  $(n_1, \dots, n_d) \in \mathbb{Z}^d$ . If  $[t]$  denotes the integer part of  $t$  and  $\{t\} = t - [t]$  is the fractional part of  $t$ , for every  $t \in \mathbb{R}$ , observe that

$$\begin{aligned}
& d(\Phi_{(t_1, \dots, t_d)}(x_1, 0, \dots, 0), \Phi_{(t_1, \dots, t_d)}(x_2, 0, \dots, 0)) \\
&= d(\Phi_{([t_1], \dots, [t_d])}(x_1, \{t_1\}, \dots, \{t_d\}), \Phi_{([t_1], \dots, [t_d])}(x_2, \{t_1\}, \dots, \{t_d\})) \\
&\leq \rho_h(\Phi_{([t_1], \dots, [t_d])}(x_1, \{t_1\}, \dots, \{t_d\}), \\
&\quad \Phi_{([t_1], \dots, [t_d])}(x_2, \{t_1\}, \dots, \{t_d\})) \\
&= \rho_h((f_1^{[t_1]} \circ \dots \circ f_d^{[t_d]}(x_1), \{t_1\}, \dots, \{t_d\}), \\
&\quad (f_1^{[t_1]} \circ \dots \circ f_d^{[t_d]}(x_2), \{t_1\}, \dots, \{t_d\})) \\
&= \sum_{\sigma \in \{0, 1\}^d} \left( \prod_{i=1}^d \sigma_i \cdot \{t_i\} + (1 - \sigma_i) \cdot (1 - \{t_i\}) \right) \\
&\quad \cdot \rho(f_1^{[t_1] + \sigma_1} \circ \dots \circ f_d^{[t_d] + \sigma_d}(x_1), f_1^{[t_1] + \sigma_1} \circ \dots \circ f_d^{[t_d] + \sigma_d}(x_2)) \\
&< \sum_{\sigma \in \{0, 1\}^d} \left( \prod_{i=1}^d \sigma_i \cdot \{t_i\} + (1 - \sigma_i) \cdot (1 - \{t_i\}) \right) \cdot \delta = \delta
\end{aligned}$$

for every  $(t_1, \dots, t_d) \in \mathbb{R}^d$ . The expansiveness condition assures that  $(x_2, 0, \dots, 0) = \Phi_{v_0}(x_1, 0, \dots, 0)$  for some  $v_0 \in \mathbb{R}^d$  such that  $\|v_0\| < \varepsilon < 1/2$ . This implies that  $x_1 = x_2$  and so the action  $\varphi : \mathbb{Z}^d \times M \rightarrow M$  is expansive.

Conversely, suppose that  $\varphi$  is expansive. In particular  $\varphi$  is also expansive with respect to the

$$\tilde{\rho}(x_1, x_2) = \min_{\sigma \in \{0, 1\}^d} \{\rho(f_1^{\sigma_1} \circ \dots \circ f_d^{\sigma_d}(x_1), f_1^{\sigma_1} \circ \dots \circ f_d^{\sigma_d}(x_2))\}$$

and let  $\zeta > 0$  be such a constant of expansiveness. Given  $\varepsilon > 0$  take  $0 < \delta < \min\{\varepsilon, \frac{1}{4}, \zeta\}$ . Suppose that  $d(\Phi_v(x_1, t_1, \dots, t_d), \Phi_{h(v)}(x_2, s_1, \dots, s_d)) < \delta$  for all  $v \in \mathbb{R}^d$  and for some continuous map  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $h(0) = 0$ . We may assume without loss of generality that  $y_1 = (x_1, 1/2, \dots, 1/2)$  and  $y_2 = (x_2, s_1, \dots, s_d)$  in the coordinates of  $M \times [0, 1]^d$  (if  $y_1$  is not in the form  $(x_1, 1/2, \dots, 1/2)$  just take  $\|w\| \leq 1/2$  such that  $\Phi_w(y_1) = (x_1, 1/2, \dots, 1/2)$  and consider the points  $\Phi_w(y_1)$  and  $\Phi_{h(w)}(y_2)$ ). Observe that

$$\tilde{\rho}(x_1, x_2) \leq d(y_1, y_2) = d(\Phi_0(y_1), \Phi_{h(0)}(y_2)) < \delta < 1/4.$$

Now, suppose that  $d(\Phi_v(y_1), \Phi_{h(v)}(y_2)) < \delta < 1/4$  for all  $v \in \mathbb{R}^d$ . In particular, taking  $v = e_i$  ( $1 \leq i \leq d$ ) it holds that  $\tilde{\rho}(f_i^n(x_1), f_i^n(x_2)) \leq d(\Phi_{n \cdot e_i}(y_1), \Phi_{h(n \cdot e_i)}(y_2)) < \delta < 1/4$  for every  $n \in \mathbb{Z}$ . Proceeding recursively, we

obtain that

$$\begin{aligned} & \tilde{\rho}((f_1^{n_1} \circ \cdots \circ f_d^{n_d})(x_1), (f_1^{n_1} \circ \cdots \circ f_d^{n_d})(x_2)) \\ & \leq d(\Phi_{(n_1, \dots, n_d)}(\overline{y_1}), \Phi_{h(n_1, \dots, n_d)}(\overline{y_2})) < \delta \end{aligned}$$

for all  $(n_1, \dots, n_d) \in \mathbb{Z}^d$ . Finally, by the expansiveness of  $\varphi$  we obtain that  $x_1 = x_2$ , implying in  $y_2 = \Phi_v(y_1)$  for some  $v \in \mathbb{R}^d$  such that  $\|v\| < \delta < \varepsilon$ .  $\square$

## 5. CENTRALIZER FOR EXPANSIVE HOMOGENEOUS $\mathbb{R}^d$ -ACTIONS

In this section first we prove that  $\mathbb{R}^d$ -actions with expansive elements have one dimensional orbits (Corollary B) and study the centralizer of homogeneous expansive  $\mathbb{R}^d$ -actions (Theorem B). In that follows,  $\|\cdot\|$  will denote the Euclidean norm in  $\mathbb{R}^d$ .

**5.1. Proof of Corollary B.** Let  $M$  be a compact Riemannian manifold and let  $\Phi : \mathbb{R}^d \times M \rightarrow M$  be a continuous action in  $M$  such that the flow  $(\Phi_{tv})_{t \in \mathbb{R}}$  is Komuro-expansive for a fixed  $v \in \mathbb{R}^d$ . Consider  $\{v, u_2, u_3, \dots, u_d\}$  a basis of  $\mathbb{R}^d$  containing the vector  $v$ .

Observe that  $\Phi_{tv} \circ \Phi_{su_i} = \Phi_{su_i} \circ \Phi_{tv}$  for all  $t, s \in \mathbb{R}$  and  $2 \leq i \leq d$ . so, by expansiveness of the flow  $(\Phi_{tv})_{t \in \mathbb{R}}$ , as a consequence of Theorem A for each  $2 \leq i \leq d$  there exists a unique function  $A_i : X \rightarrow \mathbb{R}$  invariant along orbits of flow  $(\Phi_{tv})_{t \in \mathbb{R}}$  and such that  $\Phi_{u_i}(t, x) = \Phi_v(A_i(x)t, x)$ . Consequently,

$$\Phi(t_1v + t_2u_2 + \cdots + t_du_d, x) = \Phi_v((1 + A_2(x) + \cdots + A_d(x))t, x),$$

which proves that all regular orbits of  $\Phi$  are unidimensional. This completes the proof of the corollary.

**5.2. Proof of Theorem B.** In this subsection we characterize the space of  $C^1$   $\mathbb{R}^d$ -actions  $\Psi$  that commute with an expansive  $C^1$   $\mathbb{R}^d$ -action  $\Phi$ . Our purpose is to prove that the action  $\Psi$  is a reparametrization of  $\Phi$ : there exists a continuous map  $A : M \rightarrow \mathcal{M}_{d \times d}(\mathbb{R})$  satisfying: (i)  $A(x) = A(\Phi_v(x))$  for every  $v \in \mathbb{R}^d$  and  $x \in M$ , and (ii)  $\Psi_v(x) = \Phi(A(x)v, x)$  for every  $(v, x) \in \mathbb{R}^d \times M$  (cf. Proposition 5.1 below). Since the strategy of the proof is similar to the one of Theorem A we will sketch the details and highlight the main differences. The starting point is the following canonical form for commuting vector fields, similar to the tubular neighborhood theorem.

**Lemma 5.1** (Lee [25], Theorem 18.6). *Let  $M$  be a smooth  $n$ -manifold, let  $d < n$  and let  $\Phi$  be a  $C^1$   $\mathbb{R}^d$ -action on  $M$ . Assume that  $\Phi$  is generated by smooth commuting linearly independent vector fields  $X_1, \dots, X_d$  on some open subset  $W \subseteq M$ . For each  $p \in W$  there exists an open neighborhood  $U$  of  $p$ , a  $C^1$ -diffeomorphism  $h : U \rightarrow h(U) \subset \mathbb{R}^n$  with coordinate functions  $h(q) = (s_1(q), \dots, s_n(q))$  on  $U$  and  $h(p) = 0$  and such that  $X_i = h_*^{-1} \frac{\partial}{\partial s_i}$  for  $i = 1, \dots, d$ . If  $S \subseteq U$  is an embedded codimension- $d$  submanifold and  $q$  is a point of  $S$  such that  $T_q S$  is complementary to  $\text{span}(X_1(q), \dots, X_d(q))$  then the coordinates can be chosen such that  $S$  is defined by the coordinates  $s_1 = \dots = s_d = 0$ .*

The next lemma asserts that the leaves formed by the orbits of expansive  $\mathbb{R}^d$ -actions do not admit closed curves of arbitrarily small diameter. Since the proof of the lemma is completely similar to the one of Lemma 3.2, making use of Lemma 5.1, we shall omit it.



**Lemma 5.2.** *If  $\Phi : \mathbb{R}^d \times M \rightarrow M$  is an  $C^1$  expansive homogeneous  $\mathbb{R}^d$ -action, then  $\varepsilon_0(\Phi) = \inf\{\|v\| > 0 : v \text{ is period of a periodic orbit of } \Phi\} > 0$ .*

We should observe that if an  $\mathbb{R}^d$ -action is not homogeneous then  $\varepsilon_0(\Phi)$  would be zero. Moreover, the geometry of the space of orbits in the case of non-homogeneous actions can be very complicated.

**Lemma 5.3.** *If  $\Phi : \mathbb{R}^d \times M \rightarrow M$  is an expansive  $C^1$ -action and  $\Psi \in \mathcal{Z}^1(\Phi)$  then, for all  $0 < \varepsilon < \varepsilon_0(\Phi)/3$ , there exists  $\mu > 0$  and a unique map  $z : \overline{B_\mu(0)} \times M \rightarrow B_\varepsilon(0) \subset \mathbb{R}^d$  such that  $\Psi_s(x) = \Phi(z(s, x), x)$  for all  $(s, x) \in \overline{B_\mu(0)} \times M$ . Moreover,*

- (I)  *$z$  is a continuous map,*
- (II) *If  $v, u, u + v \in \overline{B_\mu(0)}$ , then  $z(u + v, x) = z(u, x) + z(v, \Psi(u, x))$ .*

*Proof.* Although the proof is analogous to the one of Lemma 3.3, we include the construction of the local reparametrization for completeness. Let  $\varepsilon_0 > 0$  as in Lemma 5.2, take  $0 < \varepsilon < \varepsilon_0(\Phi)/3$  and let  $\delta > 0$  be given by expansiveness (recall Definition 2.4). By compactness of  $M$  there exists  $\mu > 0$  such that if  $0 < \varepsilon < \varepsilon_0/3$ , then

$$\sup_{\|u\| \leq \mu} \{d(\text{Id}, \Psi_u)\} < \delta.$$

If  $\|u\| \leq \mu$  then  $d(\Phi_v(x), \Phi_v(\Psi_u(x))) = d(\Psi_0(\Phi_v(x)), \Psi_u(\Phi_v(x))) < \delta$  for all  $(v, x) \in \mathbb{R}^d \times M$ . Since  $\Phi$  is an expansive action and  $d(\Phi_v(x), \Phi_{h(v)}(\Psi_u(x))) < \delta$  for all  $v \in \mathbb{R}^d$  (with  $h = \text{Id}$ ), there exists  $v_0 \in \mathbb{R}^d$  such that  $\Phi_{v_0}(\Psi_u(x)) = \Phi_{v_0+\eta}(x)$  for some  $\eta \in B_\varepsilon(0)$ . This implies that  $\Psi_u(x) = \Phi_\eta(x)$  for some vector  $\eta$  satisfying  $\|\eta\| < \varepsilon$ . In particular,  $\Psi_u(x)$  belongs to the orbit of  $x$  relative to the  $\mathbb{R}^d$ -action  $\Phi$ .

This defines a map  $z : \overline{B_\mu(0)} \times M \rightarrow B_\varepsilon(0)$  such that  $\Psi(u, x) = \Phi(z(u, x), x)$  for any  $(u, x) \in \overline{B_\mu(0)} \times M$ . To prove the uniqueness, observe that if  $z_1, z_2 : \overline{B_\mu(0)} \times M \rightarrow B_\varepsilon(0)$  are such that  $\Phi(z_1(u, x), x) = \Psi(u, x) = \Phi(z_2(u, x), x)$ , then  $\Phi(z_1(u, x) - z_2(u, x), x) = x$  where  $\|z_1(u, x) - z_2(u, x)\| \leq \|z_1(u, x)\| + \|z_2(u, x)\| < 2\varepsilon_0(\Phi)/3$ . Since this contradicts the existence of period smaller than  $\varepsilon_0(\Psi)$  and the uniqueness of  $z$  follows. The proof of the continuity is completely analogous to the one of Lemma 3.3 and we shall omit it.  $\square$

Next, we will construct an extension to  $\mathbb{R}^d \times M$  for the continuous reparametrization described in Lemma 5.3. More precisely we have the following:

**Lemma 5.4.** *If  $\Phi : \mathbb{R}^d \times M \rightarrow M$  is a continuous action and  $\Psi : \mathbb{R}^d \times M \rightarrow M$  is a continuous action such that for  $\mu > 0$  fixed there exists a reparametrization  $z : \overline{B_\mu(0)} \times M \rightarrow B_\varepsilon(0)$  such that  $\Psi(v, x) = \Phi(z(v, x), x)$  for any  $(v, x) \in \overline{B_\mu(0)} \times M$ , when  $0 < \varepsilon < \varepsilon_0(\Phi)/3$ , then exists a unique continuous application  $p : \mathbb{R}^d \times M \rightarrow \mathbb{R}^d$  where is extension of  $z$  and such that  $\Psi(s, x) = \Phi(p(s, x), x)$  for all  $(s, x) \in \mathbb{R}^d \times M$ .*

*Proof.* The strategy of the proof is similar to the one of Proposition 3.1, that is, to extend the local reparametrization given in Lemma 5.3 to an application  $p : \mathbb{R}^d \times M \rightarrow \mathbb{R}^d$  such that  $\Psi(s, x) = \Phi(p(s, x), x)$  for all  $(s, x) \in \mathbb{R}^d \times M$ . Due to the higher dimensional setting, there are several ways of extending the domain of the reparametrization to the euclidean space. The extension here is made radial by considering observing vectors in  $\mathbb{R}^d$  as multiples of vectors in the unit sphere  $\mathbb{S}^{d-1}$ . We shall sketch the main differences and omit some details. For this, let  $\mu > 0$  be given by Lemma 5.3, let  $N \in \mathbb{N}$  such that  $2^{-N} < \mu$  and fix  $v \in \mathbb{S}^{d-1}$ .

For each  $k \in \mathbb{N}$  let  $D_k = \{z \in \mathbb{R}^d : \frac{k}{2^N} \leq \|z\| \leq \frac{k+1}{2^N}\}$ , which contain the vectors of the form  $u = tv \in \mathbb{R}^d$  for  $t \in [k/2^N, (k+1)/2^N]$ . Now, consider the functions  $z_k : D_k \times M \rightarrow \mathbb{R}^d$  given by

$$z_k(t \cdot v, x) = z((t - k/2^N) \cdot v, x) + \sum_{i=1}^k z(1/2^N \cdot v, \Psi((t - i/2^N) \cdot v, x)), \quad (5.1)$$

for every  $k \in \mathbb{N}$ . By Lemma 5.3 and the definition of  $z_k$ , it follows that  $z_k$  is continuous and satisfies

$$z_k\left(\frac{k+1}{2^N} \cdot v, x\right) = z_{k+1}\left(\frac{k+1}{2^N} \cdot v, x\right) \quad \text{and} \quad \Psi(tv, x) = \Phi(z_k(tv, x), x) \quad (5.2)$$

for all  $x \in M$ ,  $k \in \mathbb{N}$  and  $t \in [k/2^N, (k+1)/2^N]$ . This allows to define the continuous map  $p : \mathbb{R}^d \times M \rightarrow \mathbb{R}^d$  given by

$$p(t \cdot v, x) = \begin{cases} z(t \cdot v, x), & \text{if } t \in [0, 1/2^N], v \in \mathbb{S}^{d-1} \\ z_k(t \cdot v, x), & \text{if } t \in [k/2^N, (k+1)/2^N], v \in \mathbb{S}^{d-1}, k \in \mathbb{N} \end{cases},$$

The continuity of  $p$  follows from relation (5.2) and Lemma 5.3. Moreover, since  $v \in \mathbb{S}^{d-1}$  was chosen arbitrary then Lemmas 5.3 and 5.4 imply that  $p$  satisfies  $\Psi(t \cdot v, x) = \Phi(p(t \cdot v, x), x)$  for all  $x \in M$ ,  $t \in \mathbb{R}$ ,  $v \in \mathbb{S}^{d-1}$ .

To prove the uniqueness of the reparametrization  $p$ , assume that are continuous reparametrizations  $p_1, p_2 : \mathbb{R}^d \times M \rightarrow \mathbb{R}^d$  that extend  $z$  and such that  $\Phi(p_1(u, x), x) = \Psi(u, x) = \Phi(p_2(u, x), x)$  for any  $(u, x) \in \mathbb{R}^d \times M$ . Fix  $x \in M$  and let  $\alpha_x(u) = p_1(u, x) - p_2(u, x)$ . Observe that  $\alpha_x^{-1}(0) \supset \overline{B_\mu(0)}$  and, consequently,  $\alpha_x^{-1}(0) \neq \emptyset$ . Moreover, since  $\alpha_x$  is continuous then  $\alpha_x^{-1}(0)$  is a closed subset of  $\mathbb{R}^d$ . We claim that  $\alpha_x^{-1}(0) = \mathbb{R}^d$ . If  $\alpha_x^{-1}(0) \neq \mathbb{R}^d$ , there would be a unit vector  $v \in \mathbb{S}^{d-1}$  and  $t_0 = \sup\{t > 0 : t \cdot v \in \alpha_x^{-1}(0)\} < \infty$ . Since  $\alpha_x^{-1}(0)$  is a closed subset then  $t_0 \cdot v \in \alpha_x^{-1}(0)$ . Recalling the previous discussion, setting  $x' = \Psi(p_1(t_0 \cdot v, x), x) (= \Psi(p_2(t_0 \cdot v, x), x))$ , it follows that  $\alpha_{x'}$  is identically zero in  $\overline{B_\mu(0)}$ . Thus  $\alpha_x$  is identically zero in  $\{t \cdot v : t \in [0, t_0 + \mu]\}$ , which contradicts the maximality of  $t_0$ . This proves the claim and the uniqueness of the reparametrization.  $\square$

The next lemma will complete the proof of Theorem B.

**Lemma 5.5.** *Let  $p$  be the reparametrization given by Lemma 5.4. Then  $p$  is invariant along the orbits of  $\Phi$ , that is,  $p(v, x) = p(v, \Phi(u, x))$  for all  $u, v \in \mathbb{R}^d$  and  $x \in M$ . Moreover, there exists a  $C^1$ -map  $M \ni x \mapsto A(x) \in \mathcal{M}_{d \times d}(\mathbb{R})$  so that  $p(v, x) = A(x)v$  for all  $x \in M$  and  $v \in \mathbb{R}^d$ .*

*Proof.* The arguments of Lemma 3.4 yield  $\Phi(p(v+u, x), x) = \Phi(p(v, x) + p(u, x), x)$  for every  $u, v \in \mathbb{R}^d$  and  $x \in M$ . The uniqueness of  $p$  implies that  $p(v+u, x) = p(v, x) + p(u, x)$  for all  $u, v \in \mathbb{R}^d$  and  $x \in M$ . By the later and the continuity of  $p$  we conclude that there exists a continuous map  $M \ni x \mapsto A(x) \in \mathcal{M}_{d \times d}(\mathbb{R})$  so that  $p(v, x) = A(x)v$  for all  $x \in M$  and  $v \in \mathbb{R}^d$ . The differentiability of  $A$  follows from the implicit function theorem as in the end of the proof of Theorem A. This finishes the proof of the lemma.  $\square$

In the remaining of this section we provide a geometric characterization of the linear reparametrization obtained in Theorem B, which is of independent interest in the case of a  $C^1$  and expansive  $\mathbb{R}^d$ -action on a compact manifold  $M$  of dimension

$n$  larger than  $d$ . We prove that the reparametrization can be written as a matrix of change of coordinates.

**Proposition 5.1.** *Let  $\Phi : \mathbb{R}^d \times M \rightarrow M$  a  $C^1$ -expansive and homogeneous action and  $\Psi \in \mathcal{Z}^1(\Phi)$  be such that  $\Psi(v, x) = \Phi(A(x)v, x)$  for all  $v \in \mathbb{R}^d$  and  $x \in M$ . Considering the vector fields  $X_i(\cdot) = \frac{d\Phi(te_i, \cdot)}{dt} \big|_{t=0}$  and  $Y_i(\cdot) = \frac{d\Psi(te_i, \cdot)}{dt} \big|_{t=0}$  for every  $1 \leq i \leq d$ , for each  $x \in M$  the linear map  $A(x)$  is represented by the matrix of representation of the vectors  $(Y_i(x))_{1 \leq i \leq d}$  on the basis  $(X_i(x))_{1 \leq i \leq d}$ .*

*Proof.* The homogeneity assumption assures that the vector fields  $X_1, \dots, X_d$  are linearly independent. Fix  $x \in M$ . Up to a change of coordinates we may assume without loss of generality that  $\Phi$  is (locally) an  $\mathbb{R}^d$ -action on an open set of  $\mathbb{R}^n$  ( $n = \dim M$ ) and that  $X_i(z) = e_i$  for all  $1 \leq i \leq d$  ( $< n$ ). Indeed, Lemma 5.1 guarantees that there exists an open neighborhood  $V_x \subset M$  of  $x$ , and a change of coordinates  $h : V_x \rightarrow h(V_x) \subset \mathbb{R}^n$  so that  $X_i = h_*^{-1}e_i$  for all  $1 \leq i \leq d$ . and, still denoting by  $\Phi$  the induced action on  $h_x(V)$ , one can write

$$\Phi(v, z) = z + \sum_{i=1}^d v_i \cdot X_i$$

for every  $z = h(x) \in \mathbb{R}^d$  and every small values  $(v_i)_{1 \leq i \leq d}$  such that  $\Phi(v, z)$  belongs to  $h(V_x)$ . Reducing  $V_x$  if necessary, let  $\delta > 0$  be such that  $h(V_x) = [-\delta, \delta]^n$ .

The first step in the proof of Theorem B implies that the orbits of  $\Phi$  are fixed by any element  $\Psi \in \mathcal{Z}^1(\Phi)$ . Moreover, since  $\Phi$  restricted to each of its orbits is generated by the  $d$  constant vector fields  $X_i(z) = e_i$  ( $1 \leq i \leq d$ ) then  $\Phi$  acts in each of its (local) orbits as a group of translations in  $\mathbb{R}^d$ . Indeed, in the linearization coordinates, the (local) orbit of  $z \in h(V_x)$  is  $\mathcal{F}(z) = \{w \in [-\delta, \delta]^n : w = z + te_i \text{ for some } 1 \leq i \leq d \text{ and } t \in \mathbb{R}\}$ . Then, for any given small translation vector  $g$  in  $\mathcal{F}(z) \simeq \mathbb{R}^d$  there exists a vector  $u = u_g$  such that  $\Phi(u_g, z) = z + g$ . In resume, for every  $z \in [-\delta, \delta]^n$  the map  $\Psi|_{\mathcal{F}(z)}$  commutes with all local translations in  $\mathcal{F}(z)$ .

Now we prove that in these linearization coordinates the action  $\Psi \in \mathcal{Z}^1(\Phi)$  is also a translation along the orbit  $\mathcal{F}(z)$ . Since  $\mathcal{F}(z) \simeq [-\delta, \delta]^n \subset \mathbb{R}^n$  this is an immediate consequence of the following:

**Claim:** *If  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is  $C^1$  and commutes with all translations in  $\mathbb{R}^d$  then  $f$  is itself a translation in  $\mathbb{R}^d$ .*

*Proof of the claim.* One can write in coordinate functions

$$f(x_1, x_2, \dots, x_d) = (f_1(x_1, x_2, \dots, x_d), \dots, f_d(x_1, x_2, \dots, x_d))$$

for  $(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ . Since  $f$  commutes with all the translations of form  $x + \lambda e_j$ , for  $x = (x_1, \dots, x_d)$ ,  $\lambda \in \mathbb{R}$  and where  $\{e_1, \dots, e_d\}$  denotes the canonical basis in  $\mathbb{R}^d$ , then  $f(x + \lambda e_j) = f(x) + \lambda e_j$ . Analyzing each coordinate independently, this means that  $f_i(x + \lambda e_j) = f_i(x) + \lambda \delta_{ij}$  for all  $1 \leq i, j \leq d$ , where  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  otherwise. Thus,

$$\frac{\partial f_i}{\partial x_j}(x) = \lim_{\lambda \rightarrow 0} \frac{f_i(x + \lambda e_j) - f_i(x)}{\lambda} = \delta_{ij}$$

and, consequently,  $f_i(x) = x_i + r_i$  for  $r_i = f_i(0) \in \mathbb{R}$ . This guarantees that  $f(x_1, x_2, \dots, x_d) = (x_1, x_2, \dots, x_d) + (r_1, r_2, \dots, r_d)$  is a translation and completes the proof of the claim.  $\square$

We are now in a position to complete the proof of the lemma. The previous claim implies that  $\Psi(u, y)$  is a translation for all  $(u, y) \in U$ . In particular, if  $\{u_1, \dots, u_d\}$  is a basis of  $\mathbb{R}^d$ , the flows  $(\Psi_{t \cdot u_i})_{t \in \mathbb{R}}$  are flows of translations. Indeed, by the group property of  $(\Psi_{t \cdot u_i})_{t \in \mathbb{R}}$  there exists a vector  $w_i$  such that  $\Psi_{t \cdot u_i}(x) = x + tw_i$  and consequently the  $d$  vector fields  $Y_1, Y_2, \dots, Y_d$  defining  $\Psi$  are constant. Then, one can write

$$\begin{cases} Y_1 = a_{11}X_1 + a_{21}X_2 + \dots + a_{d1}X_d \\ Y_2 = a_{12}X_1 + a_{22}X_2 + \dots + a_{d2}X_d \\ \vdots \\ Y_d = a_{1d}X_1 + a_{2d}X_2 + \dots + a_{dd}X_d \end{cases}$$

on  $U$  and let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1d} \\ a_{21} & a_{22} & \dots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \dots & a_{dd} \end{bmatrix}.$$

Then, witting  $v = \sum_{i=1}^d r_i Y_i \in \mathbb{R}^d$  as a linear combination of the vectors on the base  $(Y_i)_{1 \leq i \leq d}$ ,

$$\begin{aligned} \Psi(v, z) &= z + \sum_{j=1}^d r_j Y_j = z + \sum_{j=1}^d r_j \left[ \sum_{i=1}^d a_{ij} X_i \right] \\ &= z + \sum_{i=1}^d \left[ \sum_{j=1}^d r_j a_{ij} \right] \cdot X_i = z + Av = \Phi(Av, z) \end{aligned}$$

for all  $z \in U$ . This proves that  $A(x) = Dh(x)^{-1} A Dh(x)$  where  $A$  is the previous matrix of change of coordinates that determines the action  $\Psi$  as reparametrization of action  $\Phi$ . This finishes the proof of the lemma.  $\square$

## 6. EXAMPLES AND APPLICATIONS

This section is devoted to present some applications of our main results and a discussion on other notions of expansiveness.

**Example 6.1.** (*Suspension flows*) Given a homeomorphism  $f : M \rightarrow M$  on a compact metric space  $M$  and a continuous roof function  $r : M \rightarrow \mathbb{R}^+$  that is bounded away from zero consider the quotient space

$$M_r = \{(x, s) \in M \times \mathbb{R}^+ : 0 \leq s \leq r(x)\} / \sim$$

obtained by the equivalence relation that  $(x, r(x)) \sim (f(x), 0)$  for every  $x \in M$ . The suspension flow  $(\varphi_t)_t$  on  $M_r$  associated to  $(f, M, r)$  is defined by the “vertical displacement”  $\varphi_t(x, s) = (x, t + s)$  whenever the expression is well defined. More precisely,  $\varphi_t(x, s) = (f^k(x), t + s - \sum_{j=0}^{k-1} r(f^j(x)))$  where  $k = k(x, t, s) \in \mathbb{Z}$  is determined by  $\sum_{j=0}^{k-1} r(f^j(x)) \leq t + s < \sum_{j=0}^k r(f^j(x))$ . Clearly  $M \times \{0\}$  is a global cross-section to the flow. It follows from [11] that  $f$  is expansive if and only if the flow  $(\varphi_t)_t$  is  $C$ -expansive. In particular, from Theorem A, the suspension flow of any expansive homeomorphism (e.g. quasi-Anosov diffeomorphisms with intermittency) or Axiom A flows restricted to the non-wandering set have quasi-trivial centralizers.

**Example 6.2.** (Lorenz attractors) The Lorenz equations correspond to the system of polynomial ordinary differential equations in  $\mathbb{R}^3$

$$\begin{cases} \frac{dx}{dt} = a(y - x) \\ \frac{dy}{dt} = -xz + rx - y \\ \frac{dz}{dt} = xy - bz \end{cases}, \quad (6.1)$$

with parameters  $a, b, r \in \mathbb{R}$ . Computer simulations led Lorenz [22] to propose the existence of a “strange attractor” for the parameters  $a = 10$ ,  $b = 8/3$  and  $r = 28$ . For the classical parameters proposed by Lorenz, the three singularities of the equation (6.1) are hyperbolic, and  $\sigma_0$  belongs to the “chaotic attractor” and is accumulated by orbits of regular points. Simple computations yield that the eigenvalues of  $\sigma_0$  are  $\frac{-11 - \sqrt{1201}}{2} \approx -22,83$ ;  $-\frac{8}{3} \approx -2,67$  and  $\frac{-11 + \sqrt{1201}}{2} \approx 11,83$ . By the symmetry of the equations (6.1), the eigenvalues of  $\sigma_1$  and  $\sigma_2$  are the same. The singularity  $\sigma_1$  has a real eigenvalue  $\lambda \approx -13,85$  and two complex conjugates eigenvalues  $z, \bar{z}$  where  $z \approx 0,09 + 10,19i$ . In particular the singularities of (6.1) satisfy the non-ressonant conditions. Indeed, the singularity  $\sigma_0$  is non-ressonant since the unstable subspace is one-dimensional and the stable subspace of  $\sigma_0$  is non-ressonant because one eigenvalue is rational and the other is irrational. Finally, the singularities  $\sigma_1$  and  $\sigma_2$  are non-ressonant since their stable subspace is one-dimensional and has a pair of complex conjugate eigenvalues of along the unstable subspace.

In order to be able to describe the dynamical features of the ‘chaotic attractor’ associated to the ODE (6.1), geometric Lorenz attractors were introduced independently in [1, 15]. These form a parametrized family of vector fields, whose parameters correspond to the real eigenvalues  $\lambda_1 < \lambda_2 < 0 < -\lambda_2 < \lambda_3$  at the singularity  $\sigma_0 = (0, 0, 0)$ .

There exists a  $C^1$ -open subset of vector fields  $\mathcal{U} \subset \mathfrak{X}^\infty(\mathbb{R}^3)$  and an open ellipsoide  $V \subset \mathbb{R}^3$  containing the origin such that every  $X \in \mathcal{U}$  exhibits a geometric Lorenz attractor  $\Lambda_X = \bigcap_{t \geq 0} \overline{X_t(V)}$ , which is a partially hyperbolic attractor and whose restriction of the flow to the attractor is Komuro expansive (see e.g. [2] for precise definitions and proofs). Such construction can be performed in an open domain of a compact manifold  $M$  and if this is the case we will say that  $X \in \mathfrak{X}^1(M)$  has a geometric Lorenz attractor. Since the non-ressonance condition for the singularity  $\sigma_0$  is satisfied for both the original parameters proposed by Lorenz and is a  $C^1$ -open and  $C^\infty$  dense condition on the space of vector fields in  $\mathcal{U}$ , the following is an immediate consequence of Theorem A:

**Corollary 6.1.** Let  $\mathcal{U} \subset \mathfrak{X}^\infty(M)$  be an open set of vector fields so that every  $X \in \mathcal{U}$  has a geometric Lorenz attractor  $\Lambda_X$ . Then there exists a  $C^1$ -open and  $C^\infty$ -dense subset  $\mathcal{U}' \subset \mathcal{U}$  so that, every vector field  $X \in \mathcal{U}'$  admits a geometric Lorenz attractor  $\Lambda_X$  whose centralizer on its topological basin of attraction is trivial.

We observe that the argument used in the previous example extends to a more general class of three-dimensional flows.

**Example 6.3.** (Robustly transitive three-dimensional sets) In [28], the authors described the structure of all  $C^1$  robustly transitive sets with singularities for flows on compact Riemannian three-dimensional manifolds. These are partially hyperbolic attractors (or repellers) for the vector field with volume-expanding central

direction and have an invariant foliation whose leaves are forward contracted by the flow, and has positive Lyapunov exponent at every orbit. These are referred as singular-hyperbolic attractors or repellers. Singular-hyperbolicity is a  $C^1$ -open condition. Every singular-hyperbolic attractor is Komuro-expansive and an homoclinic class (see [2]). So Theorem A implies there exists a  $C^1$ -open and  $C^\infty$ -dense subset of  $C^\infty$  singular-hyperbolic attractors with quasi-trivial centralizer.

One should mention that the singularities of  $C^1$ -robust Komuro expansive flows are hyperbolic (cf. [24]). The following question arises naturally:

**Question 1:** Is the centralizer of Komuro expansive flows with isolated non-hyperbolic singularities trivial?

The strategy used here can probably be applied to deal with other notions of expansiveness. In [3], Artigue introduced some notions of expansiveness that we now recall. A flow is called *kinematic expansive* if for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $d(\varphi_t(x), \varphi_t(y)) < \delta$  for all  $t \in \mathbb{R}$ , then there exists  $s \in (-\varepsilon, \varepsilon)$  with  $y = \varphi_s(x)$ . A flow is *strong kinematic expansive* if every continuous reparametrization of the flow is kinematic expansive or, equivalently, all topologically equivalent flows are kinematic expansive. In [3], the author proves

$$\text{K-expansive} \Rightarrow \text{strong kinematic expansive} \Rightarrow \text{kinematic expansive.} \quad (6.2)$$

Together with (2.1), the later implies that  $C$ -expansiveness implies on kinematic expansiveness. By [3, Theorem 7.5], in the case of non-singular vector flows, the notions of  $C^1$ -robustly kinematic expansive,  $C^1$ -robustly strong kinematic expansive,  $C^1$ -robustly expansive, K-expansive or C-expansive flows coincide. Moreover, if this is the case such flows have a quasi-trivial centralizer [29]. The next example illustrates that kinematic expansive flows without singularities have quasi-trivial centralizer.

**Example 6.4.** (*Kinematic expansive flow with quasi-trivial centralizer*) Consider  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$  and the flow  $\varphi$  on  $\mathbb{T}^2$  obtained as the suspension flow of the identity map on  $\mathbb{S}^1$  by a smooth and positive smooth function  $r : \mathbb{S}^1 \rightarrow (0, +\infty)$  without any plateau. The flow  $\varphi$  is kinematic expansive but is not strong kinematic expansive. The proof of Lemma 3.3 carries on for kinematic expansive flows, which guarantees that any element  $\psi$  of the  $C^1$ -centralizer of  $\varphi$  is a locally a reparametrization of  $\varphi$ . Moreover, since  $\varphi$  has no singularities, the arguments of Subsection 3.3 and 3.4 yield that  $\psi$  is a linear reparametrization of  $\varphi$ . In other words, the centralizer of  $\varphi$  is quasi-trivial.

Clearly the previous flow can be  $C^1$ -approximated by a flow that is not kinematic expansive. In the following example we describe the centralizer of an example of strong kinematic expansive flow with a singularity.

**Example 6.5.** Consider an irrational flow on the two-dimensional torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  with vector field  $X$  and let  $f$  be any non-negative smooth function  $f$  with just one zero at some point  $p \in \mathbb{T}^2$ . The flow  $\varphi$  generated by the vector field  $fX$  is strong kinematic expansive (cf. [3, Example 2.8]). Since this flow has a non-hyperbolic singularity then Theorem A does not apply. In fact, although we do not need this here, it is not hard to show that  $\varphi$  is not even Komuro expansive. We claim that  $\mathcal{Z}^1(\varphi)$  is trivial. In fact, if  $\sigma$  the unique singularity of  $\varphi$  and  $\psi \in \mathcal{Z}^1(\varphi)$  then  $\psi_s(\sigma) = \sigma$  for all  $s \in \mathbb{R}$ . In other words,  $\sigma$  is a singularity for  $\psi$ . Moreover,



$\psi$  preserves the ( $\varphi$ -invariant) stable set  $\mathcal{B}^s(\sigma) := \{y \in \mathbb{T}^2 : d(\varphi_t(y), \sigma) \rightarrow 0 \text{ as } t \rightarrow +\infty\}$ . This set  $\mathcal{B}^s(\sigma)$  is one dimensional it is formed by the orbit of any point in  $\mathcal{B}^s(\sigma) \setminus \{\sigma\}$ . As mentioned in the previous example, the arguments used to deduce the existence and uniqueness of a continuous and  $\varphi$ -invariant function  $A : \mathbb{T}^2 \setminus \{\sigma\} \rightarrow \mathbb{R}$  so that  $\psi_t(x) = \varphi_{A(x)t}(x)$  for every  $x \in \mathbb{T}^2 \setminus \{\sigma\}$  and  $t \in \mathbb{R}$ . Now, since  $\mathcal{B}^s(\sigma)$  is dense in  $\mathbb{T}^2$  and the function  $A$  is constant along orbits of  $\varphi$  then it is constant in  $\mathbb{T}^2 \setminus \{\sigma\}$ . Thus,  $A$  clearly extends to a constant function on the torus  $\mathbb{T}^2$ , which proves that there exists  $c \in \mathbb{R}$  so that  $\psi_t(x) = \varphi_{ct}(x)$  for every  $x \in \mathbb{T}^2$  and  $t \in \mathbb{R}$ . In other words, the  $C^1$ -centralizer of  $\varphi$  is trivial.

In view of the previous example it seems natural to ask the following:

**Question 2:** Do all strong kinematic expansive with singularities have trivial centralizer?

Finally, we describe the centralizer of Anosov  $\mathbb{R}^d$ -actions.

**Example 6.6.** (*Anosov actions have quasi-trivial centralizer*) Let  $M$  be a compact Riemannian manifold and let  $\Phi : \mathbb{R}^d \times M \rightarrow M$  be an homogeneous Anosov action. Here we show that  $\Phi$  has a quasi-trivial centralizer, thus extending [18]. First we claim that every Anosov  $\mathbb{R}^d$ -action on a compact Riemannian manifold  $M$  is kinetic expansive. This is probably well known but we could not find in the literature. Let  $\mathcal{F}$  be the  $\Phi$ -orbit foliation. Then, there exists  $v \in \mathbb{R}^d$  such that the diffeomorphism  $\Phi_v$  is an Anosov element, hence normally hyperbolically. Let  $\bar{\delta} > 0$  be given by the plaque expansiveness of  $(\Phi_v, \mathcal{F})$  (recall Subsection 2.1.1). Given  $\varepsilon > 0$  let  $\delta = \min\{\varepsilon, \bar{\delta}\} > 0$  and assume that  $x, y \in M$  satisfy  $d(\Phi_u(x), \Phi_u(y)) < \delta$  for every  $u \in \mathbb{R}^d$ . In particular, the orbits of  $x, y$  by  $\Phi_v$  differ by at most  $\bar{\delta}$  (since  $d(\Phi_{nv}(x), \Phi_{nv}(y)) < \delta \leq \bar{\delta}$  for all  $n \in \mathbb{Z}$ ). Moreover, the plaque expansiveness condition implies that  $y \in \mathcal{F}(x)$ . This proves that  $y$  belongs to the orbit of  $x$  by  $\Phi$  and that  $d(x, y) < \varepsilon$ . Thus there exists a vector  $w \in \mathbb{R}^d$  such that  $\|w\| < \varepsilon$  and  $y = \Phi_w(x)$ , consequently the action  $\Phi : \mathbb{R}^d \times M \rightarrow M$  is expansive. Thus, the ingredients in the proof of Theorem B allow to conclude that  $\Phi$  has a quasi-trivial centralizer.

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WESCLEY BONOMO, UNIVERSIDADE FEDERAL DA BAHIA, AV. ADEMAR DE BARROS S/N, 40170-110 SALVADOR, BRAZIL

*E-mail address:* wescleybonomo@yahoo.com.br

JORGE ROCHA, DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DO PORTO, RUA DO CAMPO ALEGRE, 687, 4169-007 PORTO, PORTUGAL

*E-mail address:* jrocha@fc.up.pt

PAULO VARANDAS, DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DA BAHIA, AV. ADEMAR DE BARROS S/N, 40170-110 SALVADOR, BRAZIL & CMUP, UNIVERSITY OF PORTO - PORTUGAL

*E-mail address:* paulo.varandas@ufba.br

*URL:* <http://www.pgmat.ufba.br/varandas>